

Integrability conditions for space-time stochastic integrals: theory and applications

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March 13, 2013

Abstract

We study stochastic integrals taken over time and space driven by a random measure. Integrability conditions are given based on predictable characteristics of the random measure. We apply our conditions to a variety of examples, in particular to ambit processes, which represent a rich model class.

AMS 2010 Subject Classifications: primary: 60G48, 60G57, 60H05
secondary: 60G10, 60G22

Keywords: ambit processes, integrability conditions, Lévy basis, predictable characteristics, random measure, stochastic integration, supCARMA, superposition of COGARCH processes, supOU

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1 Introduction

Since Itô introduced the stochastic integral in his famous paper [Itô44], the theory of finite dimensional stochastic integration was brought to maturity mainly by the French school during the 1970s and 1980s. Based on the fundamental Bichteler-Dellacherie theorem ([Bic79, Mey79]) an extension to the infinite dimensional case, namely stochastic integrals driven by random measures, was the next consequent step ([BJ83]). See also [Bic02] for a full account of this stochastic integration theory.

Due to the overwhelming interest in pricing and hedging of financial products, Itô's integration theory has been in the focus of applications during the past 20 years. The introduction of a stochastic ambit process

$$Y(t, x) := \int_{A(t, x)} h(t, x, s, \xi) \sigma(s, \xi) \Lambda(ds, d\xi), \quad t \in \mathbb{R}, x \in \mathbb{R}^d, \quad (1.1)$$

for modelling physical space-time phenomena like turbulence as a stochastic space-time process (cf. [BNS04]) revived interest in the definition and properties of stochastic integrals driven by random measures. However, applications of such integrals go far beyond turbulence modelling. Stochastic processes in finance based on a Heath-Jarrow-Morton approach also rely on a spatial structure like forward contracts in bond or electricity markets, or volatility surfaces in option markets (cf. [BNBV13, ABKZ12, Hom11]). Further applications of ambit processes have been suggested for instance for brain imaging ([JRNMJ13]) and tumor growth ([JJSBN07, BNS07]).

The concept of an ambit process has also been successfully invoked to define superpositions of stochastic processes like Ornstein-Uhlenbeck processes ([BN01, BNS11]). As we shall show below, this approach can be extended to the larger class of CARMA processes ([Bro01]). Furthermore, in [BCK13] ambit processes have been used to define superpositions of COGARCH processes, which have been introduced in [KLM04]; see also [KMS11]. In its simplest case superposition leads to multi-factor models, extensions of the one-factor models, which has become indispensable by the fact that all prominent continuous-time one-factor stochastic volatility models, linear or non-linear, exhibit a deterministic functional relationship between jumps in the price and jumps in the volatility processes; cf. [JKM12]. As we shall see, the supCOGARCH model needs the full ambit model definition and stochastic integration theory, since for this model the volatility σ and the random measure Λ are no longer independent.

Our paper is organised as follows. Section 2 introduces the notation, the concept of a random measure, the construction of the stochastic integral, and first properties of the integral. Section 3 derives a canonical decomposition as known for semimartingales. Section 4 presents our main results providing integrability conditions in terms of the canonical characteristics from Section 3. An explicit construction of random measures as product measures is given in Section 5, where we prove Fubini's theorem for random measures. Section 6 is dedicated to examples and again emphasizes the relevance of our integrability conditions.

2 Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, P)$ be a stochastic basis satisfying the usual assumptions of completeness and right-continuity. Denote the base space by $\bar{\Omega} := \Omega \times \mathbb{R}$ and the optional (resp. predictable) σ -field on $\bar{\Omega}$ by \mathcal{O} (resp. \mathcal{P}). Furthermore, fix some Lusin space E , equipped with its Borel σ -field \mathcal{E} . Using the abbreviations $\tilde{\Omega} := \Omega \times \mathbb{R} \times E$ and $\tilde{\mathcal{O}} := \mathcal{O} \otimes \mathcal{E}$ (resp. $\tilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{E}$), we call a function $H: \tilde{\Omega} \rightarrow \mathbb{R}$ **optional** (resp. **predictable**) if it is $\tilde{\mathcal{O}}$ -measurable (resp. $\tilde{\mathcal{P}}$ -measurable). We will often use the symbols \mathcal{O} and \mathcal{P} (resp. $\tilde{\mathcal{O}}$ and $\tilde{\mathcal{P}}$) also for the collection of optional and predictable functions from $\bar{\Omega}$ (resp. $\tilde{\Omega}$) to \mathbb{R} . For simplicity we write $A_t := A \cap (\Omega \times (-\infty, t])$ (resp. $A_t := A \cap (\Omega \times (-\infty, t] \times E)$) for sets $A \in \mathcal{P}$ (resp. $A \in \tilde{\mathcal{P}}$). Furthermore, if μ is a signed measure and X a finite variation process, we write $|\mu|$ and $|X|$ for the variation of μ and the variation process of X , respectively. Finally, let $L^p = L^p(\Omega, \mathcal{F}, P)$, $p \in [0, \infty)$, and set as usual

$$\|X\|_p := \mathbb{E}[|X|^p]^{1/p}, \quad p \geq 1, \quad \|X\|_p := \mathbb{E}[|X|^p], \quad 0 < p < 1, \quad \|X\|_0 := \mathbb{E}[|X| \wedge 1]$$

for $X \in L^p$. Among several definitions of a random measure in the literature, the following two are the most frequent ones: in its essence, a random measure is either a random variable whose realisations are measures on some measurable space (e.g. [Kal86, JS03]) or it is a σ -additive set function with values in the space L^p (e.g. [BJ83, RR89, MP77, KP96]). We will call the first type a *strict random measure* and the second type an *L^p -valued random measure*. More precisely, we have:

Definition 2.1 Let $(\tilde{O}_k)_{k \in \mathbb{N}}$ be a sequence of sets in $\tilde{\mathcal{P}}$ with $\tilde{O}_k \uparrow \tilde{\Omega}$. Set $\tilde{\mathcal{P}}_M := \bigcup_{k=1}^{\infty} \mathcal{P}|_{\tilde{O}_k}$, which is the collection of all sets $A \in \tilde{\mathcal{P}}$ such that $A \subseteq \tilde{O}_k$ for some $k \in \mathbb{N}$. An **L^p -valued σ -finite random measure** on $\mathbb{R} \times E$ is a mapping $M: \tilde{\mathcal{P}}_M \rightarrow L^p$ satisfying:

- (1) $M(\emptyset) = 0$ a.s.,
- (2) For any sequence $(A_i)_{i \in \mathbb{N}}$ of pairwise disjoint sets in $\tilde{\mathcal{P}}_M$ with $\bigcup_{i=1}^{\infty} A_i \in \tilde{\mathcal{P}}_M$ we have

$$M\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} M(A_i) \quad \text{in } L^p.$$

- (3) For all $A \in \tilde{\mathcal{P}}_M$ with $A \subseteq \tilde{\Omega}_t$ for some $t \in \mathbb{R}$ we have $M(A) \in \mathcal{F}_t$.
- (4) For all $A \in \tilde{\mathcal{P}}_M$, $t \in \mathbb{R}$ and $F \in \mathcal{F}_t$, we have

$$M(A \cap (F \times (t, \infty) \times E)) = 1_F M(A \cap (\Omega \times (t, \infty) \times E)).$$

If $p = 0$, we only say **random measure**; if \tilde{O}_k can be chosen as $\tilde{\Omega}$ for all $k \in \mathbb{N}$, M is called a **finite** random measure; and finally, if E consists of only one point, M is called a **null-spatial** random measure.

We want to point out three particularly important examples:

Example 2.2

- (1) In the null-spatial case, there is a one-to-one correspondence between L^0 -valued σ -finite random measures and the class of formal semimartingales introduced by [Sch81]. Integration theory in this null-spatial case has been revisited in [BOGP13].
- (2) A **strict random measure** is a signed transition kernel $\mu(\omega, dt, dx)$ from (Ω, \mathcal{F}) to $(\mathbb{R} \times E, \mathcal{B}(\mathbb{R}) \otimes \mathcal{E})$ with the following properties:
 - (a) There is a strictly positive function $V \in \tilde{\mathcal{P}}$ such that the random variable $\int_{\mathbb{R} \times E} V(t, x) |\mu|(dt, dx)$ is in L^1 .
 - (b) For $\tilde{\mathcal{O}}$ -measurable functions W such that W/V is bounded, the process

$$W * \mu_t := \int_{(-\infty, t] \times E} W(s, x) \mu(ds, dx), \quad t \in \mathbb{R},$$

is optional.

If μ is a positive transition kernel, this notion coincides with the notion of an optional $\tilde{\mathcal{P}}$ - σ -finite random measure in the sense of [JS03], Chapter II. Also the predictable compensator of a strict random measure is defined there. In the following we will frequently use their notation and results on strict random measures. Obviously, a strict random measure is an L^0 -valued σ -finite random measure. For more details on that, see also [BJ83], Examples 5 and 6.

- (3) Another important example are **independently scattered infinitely divisible random measures**, also called **Lévy bases**. See Section 6.1 for a detailed discussion.

We want to recall the definition of stochastic integrals w.r.t. random measures and follow [Bic02, BJ83, BOGP13] very closely. Let \mathcal{S}_M denote the collection of **simple integrands**, which are functions $\tilde{\Omega} \rightarrow \mathbb{R}$ of the form

$$S := \sum_{i=1}^r a_i 1_{A_i}, \quad r \in \mathbb{N}, a_i \in \mathbb{R}, A_i \in \tilde{\mathcal{P}}_M. \quad (2.1)$$

Naturally the stochastic integral of S w.r.t. M is defined by

$$\int S dM := \sum_{i=1}^r a_i M(A_i). \quad (2.2)$$

Now defining \mathcal{S}_M^\uparrow as the collection of functions $\tilde{\Omega} \rightarrow \mathbb{R}$ which are positive and can be written as the pointwise supremum of simple integrands, we define the **Daniell mean** $\|\cdot\|_{M,p}^D: \mathbb{R}^{\tilde{\Omega}} \rightarrow [0, \infty]$ by

- $\|K\|_{M,p}^D := \sup_{S \in \mathcal{S}_M, |S| \leq K} \left\| \int S dM \right\|_p$, if $K \in \mathcal{S}_M^\uparrow$, and
- $\|H\|_{M,p}^D := \inf_{K \in \mathcal{S}_M^\uparrow, |H| \leq K} \|K\|_{M,p}^D$ for arbitrary functions $H: \tilde{\Omega} \rightarrow \mathbb{R}$.

The next lemma plays an important role in Daniell's extension procedure:

Lemma 2.3 Let $(S_n)_{n \in \mathbb{N}} \subseteq \mathcal{S}_m$ decrease pointwise to 0. Then $\int S_n dM \rightarrow 0$ in L^p .

The proof is completely analogous to the null-spatial case since the structure of the underlying base space ($\tilde{\Omega}$ or just $\tilde{\Omega}$) is not important. In this case, the proof is given in [BOGP13] for $p = 0$. The proof also works for $p > 0$, given the fact that Theorem A.1.3 in [KW92] holds and that L^p -valued σ -finite random measures are bounded on each \tilde{O}_k , $k \in \mathbb{N}$ (see [FS76], Theorem 2).

By Lemma 2.3 we are now able to apply Daniell's extension procedure, which is carried out in [Bic02], Chapter 3, in full detail. The most important steps and results are the following: a function $H: \tilde{\Omega} \rightarrow \mathbb{R}$ is called **integrable** w.r.t. M if there is a sequence of simple integrands $(S_n)_{n \in \mathbb{N}}$ such that $\|H - S_n\|_{M,p}^D \rightarrow 0$ as $n \in \mathbb{N}$. Then the **stochastic integral** of H w.r.t. M defined by

$$\int H dM := \int_{\mathbb{R} \times E} H(t, x) M(dt, dx) := \lim_{n \rightarrow \infty} \int S_n dM \quad (2.3)$$

exists in L^p and does not depend on the choice of $(S_n)_{n \in \mathbb{N}}$. The collection of integrable functions is denoted by $L^{1,p}(M)$ and can be characterized as follows (see [Bic02], Theorem 3.4.10 and Theorem 3.2.24):

Theorem 2.4 Let $F^{1,p}(M)$ be the collection of functions H with $\|rH\|_{M,p}^D \rightarrow 0$ as $r \rightarrow 0$. If we identify two functions coinciding up to a set whose indicator function has Daniell mean 0, then

$$L^{1,p}(M) = \tilde{\mathcal{P}} \cap F^{1,p}(M). \quad (2.4)$$

Theorem 2.5 (Dominated convergence theorem, DCT)

Let $(H_n)_{n \in \mathbb{N}}$ be a sequence in $L^{1,p}(M)$ converging pointwise to some limit H . If there exists some function $F \in F^{1,p}(M)$ with $|H_n| \leq F$ for each $n \in \mathbb{N}$, both H and H_n are integrable with $\|H - H_n\|_{M,p}^D \rightarrow 0$ as $n \rightarrow \infty$ and

$$\int H dM = \lim_{n \rightarrow \infty} \int H_n dM \quad \text{in } L^p. \quad (2.5)$$

Given a predictable function $H \in \tilde{\mathcal{P}}$, we can obviously define a new random measure HM in the following way:

$$K \in L^{1,0}(HM) :\Leftrightarrow KH \in L^{1,0}(M), \quad \int K d(HM) := \int KH dM. \quad (2.6)$$

This indeed defines a random measure provided there exists a sequence $(\tilde{O}_k)_{k \in \mathbb{N}} \subseteq \tilde{\mathcal{P}}$ with $\tilde{O}_k \uparrow \tilde{\Omega}$ and $1_{\tilde{O}_k} \in L^{1,0}(HM)$ for all $k \in \mathbb{N}$. But this construction does not extend the class $L^{1,0}(M)$ of integrable functions w.r.t. M . However, as shown in [BJ83], §3, we can define stochastic integrals for a larger class of integrands in the following way: Given an L^p -valued σ -finite random measure M , fix some $\tilde{\mathcal{P}}$ -measurable function H such that:

$$\text{There exists a predictable process } K: \tilde{\Omega} \rightarrow \mathbb{R}, K > 0, \text{ such that } KH \in L^{1,p}(M). \quad (2.7)$$

Now set $\bar{O}_k := \{K \geq k^{-1}\}$ for $k \in \mathbb{N}$, which obviously defines predictable sets increasing to $\bar{\Omega}$, and then $\mathcal{P}_{H \cdot M} := \{A \in \mathcal{P} : A \subseteq \bar{O}_k \text{ for some } k \in \mathbb{N}\}$. Then

$$H \cdot M : \mathcal{P}_{H \cdot M} \rightarrow L^p, (H \cdot M)(A) := \int 1_A H \, dM,$$

clearly defines a null-spatial L^p -valued σ -finite random measure. The following proposition is known from [BJ83] and proved by [BOGP13], Theorem 2.4, in the null-spatial case. We include its proof for the sake of completeness.

Proposition 2.6 In the above set-up the following holds:

(1) If $H \in L^{1,p}(M)$, $H \cdot M$ is an L^p -valued finite random measure and we have

$$\int 1 \, d(H \cdot M) = \int H \, dM.$$

(2) If $G : \bar{\Omega} \rightarrow \mathbb{R}$ is a predictable process, we have $G \in L^{1,p}(H \cdot M)$ if and only if $\|rGH\|_{M,p} \rightarrow 0$ as $r \rightarrow 0$, where for any $\tilde{\mathcal{P}}$ -measurable function we set

$$\|H\|_{M,p} := \sup_{\substack{F : \bar{\Omega} \rightarrow \mathbb{R} \text{ predictable,} \\ |F| \leq 1, FH \in L^{1,p}(M)}} \left\| \int FH \, dM \right\|_p. \quad (2.8)$$

In this case we have $\int G \, d(H \cdot M) = \int GH \, dM$.

Proof.

(1) Since $H \in L^{1,p}(M)$, we have

$$\|r \cdot 1\|_{H \cdot M, p}^D = \sup_{S \in \mathcal{S}_{H \cdot M}, |S| \leq 1} \left\| r \int SH \, dM \right\|_p \leq \|rH\|_{M,p}^D \rightarrow 0.$$

(2) The statement is valid because

$$\begin{aligned} \|G\|_{H \cdot M, p}^D &= \sup_{S \in \mathcal{S}_{H \cdot M}, |S| \leq |G|} \left\| \int SH \, dM \right\|_p = \sup_{\substack{F : \bar{\Omega} \rightarrow \mathbb{R} \text{ predictable,} \\ |F| \leq |G|, FH \in L^{1,p}(M)}} \left\| \int FH \, dM \right\|_p \\ &= \sup_{\substack{F : \bar{\Omega} \rightarrow \mathbb{R} \text{ predictable,} \\ |F| \leq 1, FGH \in L^{1,p}(M)}} \left\| \int FGH \, dM \right\|_p = \|GH\|_{M,p}. \end{aligned}$$

Note that in the second equality, the \leq -part is clear. For the \geq -part, choose a sequence $(S_n)_{n \in \mathbb{N}} \subseteq \mathcal{S}_{H \cdot M}$ with $S_n \rightarrow S$ pointwise and $|S_n| \leq |F|$ for all $n \in \mathbb{N}$. Then $S_n H$ converges to FH pointwise and the DCT 2.5 applies since $FH \in L^{1,p}(M)$.

□

In view of Proposition 2.6 it makes sense to call any $\tilde{\mathcal{P}}$ -measurable function H with (2.7) **integrable** w.r.t. M , written $H \in L^p(M)$, if $H \cdot M$ is an L^p -valued finite random measure. In this case set

$$\int H \, dM := (H \cdot M)(\bar{\Omega}).$$

By Proposition 2.6 we have

$$L^p(M) = \{H \in \tilde{\mathcal{P}} : H \text{ satisfies (2.7) and } \|rH\|_{M,p} \xrightarrow{r \rightarrow 0} 0\}. \quad (2.9)$$

In the null-spatial case we have $L^{1,0}(M) = L^0(M)$. But in the general case the inclusion $L^{1,p}(M) \subseteq L^p(M)$ is strict. See [BJ83], §3b, and the references therein for examples.

In the sequel we will frequently use the following fact (see [BOGP13], Example 4.1): If M is a finite random measure, the process $(M(\tilde{\Omega}_t))_{t \in \mathbb{R}}$ has a càdlàg modification, which is then a semimartingale up to infinity w.r.t. to the underlying filtration. This semimartingale will be identified with M and will be written as $(M_t)_{t \in \mathbb{R}}$.

3 Predictable characteristics of random measures

Our aim is to decompose a random measure M in a way which is analogous to the canonical decomposition of semimartingales (cf. [JS03], Theorem 2.34 of Chapter II). In the general case, results can be found in [BJ83], §4c, and [Bic02], Chapters 4.3 and 4.4. If $p \geq 2$, one can also consult [Leb95a], Theorem 1. To obtain more structure in the case $p = 0$, we have to impose some restrictions on M :

Definition 3.1 Let M be a random measure where \tilde{O}_k can be chosen as $O_k \times E_k$ with $O_k \uparrow \bar{\Omega}$ and $E_k \uparrow E$. Set $\mathcal{E}_M := \bigcup_{k=1}^{\infty} \mathcal{E}|_{E_k}$.

- (1) M is called **spatially independently scattered** if for all pairs of disjoint sets $U_1, U_2 \in \mathcal{E}_M$ and $k \in \mathbb{N}$ the semimartingales $1_{O_k \times U_i} \cdot M$, $i = 1, 2$, are independent.
- (2) M has **no fixed time of discontinuity** if for all $U \in \mathcal{E}_M$ and $k \in \mathbb{N}$ the semimartingale $X^k := 1_{O_k \times U} \cdot M$ satisfies $\Delta X_t^k = 0$ a.s. for each $t \in \mathbb{R}$.
- (3) M has **different times of discontinuity** if for all $k \in \mathbb{N}$ and disjoint sets $U_1, U_2 \in \mathcal{E}_M$ the semimartingales $1_{O_k \times U_i} \cdot M$, $i = 1, 2$, a.s. never jump at the same time.

Obviously, the first two properties together imply the third one. In this case, we have the following very simple lemma which will be used later:

Lemma 3.2 Let X^1 and X^2 be two semimartingales up to infinity which a.s. have different jump times. Furthermore, let $\tau: \mathbb{R} \rightarrow \mathbb{R}$ be a truncation function, which is a bounded function with $\tau(y) = y$ in a neighbourhood of 0. If (B^1, C^1, ν^1) and (B^2, C^2, ν^2) denote the semimartingale characteristics of X^1 and X^2 w.r.t. τ , the characteristics (B, C, ν) of $X = X^1 + X^2$ w.r.t. τ are given by $(B^1 + B^2, C^1 + C^2, \nu^1 + \nu^2)$.

Proof. By assumption, we have

$$\{\Delta X(\omega) \neq 0\} = \{\Delta X^1(\omega) \neq 0\} \dot{\cup} \{\Delta X^2(\omega) \neq 0\}.$$

It follows that, almost surely, the jump measure μ of X is the sum of the jump measures μ^1 and μ^2 of X^1 and X^2 , which directly results in $\nu = \nu^1 + \nu^2$. Since by construction, B is the finite variation process in the canonical decomposition of the special semimartingale $X - (y - \tau(y)) * \mu$, we also obtain $B = B^1 + B^2$. Since $[X^{1,c}, X^{2,c}] = 0$ by independence, we also have $X^c = X^{1,c} + X^{2,c}$, which implies $C = C^1 + C^2$. \square

We are now able to give a canonical decomposition for random measures generalizing the results of [JS03] and [BOGP13]. We write $\mathcal{B}_0(\mathbb{R})$ for the collection of Borel sets on \mathbb{R} which are bounded away from 0. Furthermore, if X is a semimartingale up to infinity, we write $\mathfrak{B}(X)$ for its first characteristic, $[X]$ for its quadratic variation, X^c for its continuous part (all of them starting at $-\infty$ with 0), μ^X for its jump measure and ν^X for its predictable compensator. Finally, if $U \in \mathcal{E}$, $M|_U$ denotes the random measure given by $M|_U(A) = M(A \cap (\bar{\Omega} \times U))$ for $A \in \tilde{\mathcal{P}}_M$.

Theorem 3.3 Let M be a spatially independently scattered random measure with different times of discontinuity. Then the mappings

$$B(A) := \mathfrak{B}(1_A \cdot M)_\infty, \quad M^c(A) := (1_A \cdot M)_\infty^c, \quad C(A) := [(1_A \cdot M)^c]_\infty, \quad A \in \tilde{\mathcal{P}}_M,$$

are random measures and

$$\mu(A, V) := \mu^{1_A \cdot M}(\mathbb{R} \times V), \quad \nu(A, V) := \nu^{1_A \cdot M}(\mathbb{R} \times V), \quad A \in \tilde{\mathcal{P}}_M, V \in \mathcal{B}_0(\mathbb{R}), \quad (3.1)$$

can be extended to random measures on $\tilde{\mathcal{P}}_M \otimes \mathcal{B}_0(\mathbb{R})$.

Moreover, (B, C, ν) can be chosen as predictable strict random measures and are uniquely determined by M . They are called the **characteristic triplet** of M .

Furthermore, let $A \in \tilde{\mathcal{P}}_M$ and fix a truncation function τ . Then $1_A(t, x)(y - \tau(y))$ (resp. $1_A(t, x)\tau(y)$) is integrable w.r.t. μ (resp. $\mu - \nu$), and we have

$$\begin{aligned} M(A) &= B(A) + M^c(A) + \int_{\mathbb{R} \times E \times \mathbb{R}} 1_A(t, x)(y - \tau(y)) \mu(dt, dx, dy) + \\ &\quad + \int_{\mathbb{R} \times E \times \mathbb{R}} 1_A(t, x)\tau(y) (\mu - \nu)(dt, dx, dy), \end{aligned} \quad (3.2)$$

Before proving this theorem, we recall some results related to the semimartingale topology introduced by [Eme79]. On the space \mathcal{SM} of semimartingales X up to infinity we can introduce a topology via

$$\|X\|_{\mathcal{SM}} := \sup_{|H| \leq 1, H \in \mathcal{P}} \left\| \int_{-\infty}^{\infty} H_t dX_t \right\|_0.$$

Under this topology we have the following continuity results:

Lemma 3.4

- (1) Let $(X^n)_{n \in \mathbb{N}} \subseteq \mathcal{SM}$ and (B^n, C^n, ν^n) denote the semimartingale characteristics of X^n . If $X^n \rightarrow 0$ in \mathcal{SM} , then each of the following semimartingales converges to 0 in \mathcal{SM} as well: B^n , $X^{c,n}$, C^n , $[X^n]$, $(y - \tau(y)) * \mu^n$ and $\tau(y) * (\mu^n - \nu^n)$.

- (2) If $W(\omega, t, y)$ is a positive bounded predictable function, then $W * \mu^n \rightarrow 0$ in probability if and only if $W * \nu^n \rightarrow 0$ in probability. Similarly, $W * \mu^n < \infty$ a.s. if and only if $W * \nu^n < \infty$ a.s.
- (3) The collection of predictable finite variation processes is closed under the semimartingale topology.

For the first part of this lemma, read along the lines of the proof of Theorem 4.10 in [BOGP13] and see [Eme79], p. 276. The second part is taken from [BOGP13], Lemma 4.8 and 4.12. The third claim is proved in [Mém80], Théorème IV.7.

Now we are able to prove Theorem 3.3:

Proof. We first show that for fixed $k \in \mathbb{N}$, the mapping $(S, U) \mapsto B(S \times U)$ is a finitely additive set function from the semiring $\mathcal{H} := \mathcal{P}|_{O_k} \times \mathcal{E}|_{E_k}$ to L^0 : Let $(S_n \times U_n)_{n=1}^N$ be finitely many pairwise disjoint sets in \mathcal{H} whose union is again in \mathcal{H} . By induction, we can assume that the statement holds in the case of $N - 1$ or fewer such sets. Now there are two trivial cases: if there are two equal sets among $(U_n)_{n=1}^N$, the statement follows from the induction hypothesis; if $(U_n)_{n=1}^N$ are pairwise disjoint sets, the finite additivity of B follows from Lemma 3.2. In the remaining cases we may assume w.l.o.g. that there is some $i \in \{1, \dots, N - 1\}$ with $U_i \setminus U_N \neq \emptyset$. Then

$$\begin{aligned}
 B\left(\bigcup_{n=1}^N S_n \times U_n\right) &= \mathfrak{B}\left(1_{\bigcup_{n=1}^N S_n \times U_n} \cdot M\right)_\infty = \mathfrak{B}\left(\sum_{n=1}^N (1_{S_n \times U_n} \cdot M)\right)_\infty \\
 &= \mathfrak{B}\left(\sum_{n=1}^N (1_{S_n \times U_n} \cdot M|_{U_N}) + \sum_{n=1}^N (1_{S_n \times U_n} \cdot M|_{U_N^c})\right)_\infty \\
 &= \mathfrak{B}\left(\sum_{n=1}^N (1_{S_n \times U_n} \cdot M|_{U_N})\right)_\infty + \mathfrak{B}\left(\sum_{n=1}^{N-1} (1_{S_n \times U_n} \cdot M|_{U_N^c})\right)_\infty \\
 &= \sum_{n=1}^N \mathfrak{B}(1_{S_n \times U_n} \cdot M|_{U_N})_\infty + \sum_{n=1}^{N-1} \mathfrak{B}(1_{S_n \times U_n} \cdot M|_{U_N^c})_\infty = \sum_{n=1}^N B(S_n \times U_n).
 \end{aligned}$$

Here equality in the third line and in the last line is again due to Lemma 3.2. In the forth line we have used the induction hypothesis for both summands: For the second one this is obviously allowed; for the first one, pick some $x \in U_i \setminus U_N$ and $\bar{\omega} \in S_N$. As $\bigcup_{n=1}^N S_n \times U_n$ lies in \mathcal{H} again, $(\bar{\omega}, x)$ must be in $S_j \times U_j$ for some $j = 1, \dots, N - 1$. But then it follows that $U_j \cap U_N = \emptyset$ and we may apply the induction hypothesis because otherwise there exists some $y \in U_j \cap U_N$ such that $(\bar{\omega}, y) \in S_j \times U_j$ and $(\bar{\omega}, y) \in S_N \times U_N$, contradicting the disjointness of these two sets.

The next step is to extend the set function B on \mathcal{H} to the ring generated by \mathcal{H} , which is exactly

$$\mathcal{R}(\mathcal{H}) = \left\{ \bigcup_{n=1}^N C_n : N \in \mathbb{N}, C_n \in \mathcal{H} \text{ pairwise disjoint} \right\}.$$

Putting $B(\bigcup_{n=1}^N C_n) := \sum_{n=1}^N B(C_n)$, we obtain a well-defined extension of B to $\mathcal{R}(\mathcal{H})$, which can be shown to be consistent with the original definition of B and still finitely

additive. Furthermore, $\mathcal{R}(\mathcal{H})$ is an algebra since it contains $O_k \times E_k$. Hence we can extend the set function B further to measures on $\sigma(\mathcal{H}) = \tilde{\mathcal{P}}|_{\tilde{O}_k}$: using [KW92], Theorem B.1.1, we only have to show:

$$(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{R}(\mathcal{H}) \text{ with } \limsup_{n \rightarrow \infty} A_n = \emptyset \implies \lim_{n \rightarrow \infty} B(A_n) = 0 \text{ in } L^0. \quad (3.3)$$

In fact, under the assumption on the left-hand side of (3.3), $1_{A_n} \cdot M \rightarrow 0$ in \mathcal{SM} :

$$\begin{aligned} \|1_{A_n} \cdot M\|_{\mathcal{SM}} &= \sup_{|H| \leq 1, H \in \mathcal{P}} \left\| \int H d(1_{A_n} \cdot M) \right\|_0 = \sup_{|H| \leq 1, H \in \mathcal{P}} \left\| \int H 1_{A_n} dM \right\|_0 \\ &\leq \sup_{S \in \mathcal{S}_M, |S| \leq 1_{A_n}} \left\| \int S dM \right\|_0 = \|1_{A_n}\|_{M,0}^D \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

due to the DCT 2.5 with $1_{O_k \times E_k}$ as dominating function. Using Lemma 3.4(1), Equation (3.3) follows.

This extension still coincides with the definition of B in Theorem 3.3: From the construction given in the proof of [KW92], Theorem B.1.1., we know that given $A \in \tilde{\mathcal{P}}|_{\tilde{O}_k}$, there is a sequence of sets $(A_n)_{n \in \mathbb{N}}$ in $\mathcal{R}(\mathcal{H})$ with $\limsup((A \setminus A_n) \cup (A_n \setminus A)) = \emptyset$ and $B(A_n) \rightarrow B(A)$ in L^0 with $n \rightarrow \infty$. As above we obtain $1_{A_n} \cdot M \rightarrow 1_A \cdot M$ in \mathcal{SM} , which implies the assertion. And of course, B is unique and $B(A)$ does not depend on the choice of $k \in \mathbb{N}$ with $A \subseteq O_k$.

Finally, we prove that B corresponds to a predictable strict random measure. By Theorem 4.10 in [BJ83] it suffices to show that for $H \in L^{1,0}(B)$ the semimartingale $H \cdot B$ is predictable and has finite variation on bounded intervals. If $H \in \mathcal{S}_M$, this follows from linearity and the fact that the first characteristic of a semimartingale up to infinity is a predictable finite variation process. In the general case choose a sequence $(S_n)_{n \in \mathbb{N}} \subseteq \mathcal{S}_M$ with $S_n \rightarrow H$ pointwise and $|S_n| \leq H$ for all $n \in \mathbb{N}$. As $n \rightarrow \infty$, we have $S_n \cdot B \rightarrow H \cdot B$ in \mathcal{SM} due to the DCT 2.5. By Lemma 3.4(3) we conclude that also $H \cdot B$ is a predictable finite variation process.

The same procedure also applies to C , and for M^c one can use [BJ83], Theorem 4.13. Let us proceed to μ and ν , where in both cases we first fix some $V \in \mathcal{B}_0(\mathbb{R})$ with $\inf\{|x| : x \in V\} \geq \epsilon > 0$ and $\epsilon < 1$. Then, in order to apply the same construction scheme as for B , only the proof of (3.3) for μ and ν is different. To this end, let $(A_n)_{n \in \mathbb{N}}$ be as on the left-hand side of (3.3), which means that we again have $1_{A_n} \cdot M \rightarrow 0$ in \mathcal{SM} . Now define $\tilde{\tau}(y) = (y \wedge \epsilon) \vee (-\epsilon)$ and choose $K > 1$ such that $|\tilde{\tau}(y)| \leq K(y^2 \wedge 1)$ for $|y| \geq \epsilon$. Then

$$\begin{aligned} \|\mu(A_n, V)\|_0 &= \left\| \frac{1_V(y)}{|\tilde{\tau}(y)|} |\tilde{\tau}(y)| * \mu_{\infty}^{1_{A_n} \cdot M} \right\|_0 \leq \epsilon^{-1} \|1_V(y) |\tilde{\tau}(y)| * \mu_{\infty}^{1_{A_n} \cdot M}\|_0 \\ &\leq K \epsilon^{-1} \|(y^2 \wedge 1) * \mu_{\infty}^{1_{A_n} \cdot M}\|_0 \leq K \epsilon^{-1} \|[1_{A_n} \cdot M]_{\infty}\|_0 \rightarrow 0, \end{aligned}$$

where the last step follows from Lemma 3.4(1). Part (2) of the same lemma yields that also $\nu(A_n, V) \rightarrow 0$ in L^0 as $n \rightarrow \infty$. As a consequence, we can use [BJ83], Theorem 4.12, to show that $\mu(\cdot, V)$ and $\nu(\cdot, V)$ can be identified as positive strict random measures. Observing that $\mu(A, \cdot)$ (resp. $\nu(A, \cdot)$) is clearly also a positive (and predictable) strict random

measure for given $A \in \tilde{\mathcal{P}}_M$, μ (resp. ν) can be extended to a positive (and predictable) strict random measure on the product $\tilde{\mathcal{P}}_M \otimes \mathcal{B}_0(\mathbb{R})$ (see [RR89], Proposition 2.4). Of course, ν is the predictable compensator of μ .

The integrability of $1_A(t, x)(y - \tau(y))$ (resp. $1_A(t, x)\tau(y)$) w.r.t. μ (resp. $\mu - \nu$) is an obvious consequence of (3.1) and the corresponding statements in the null-spatial case. Finally, the canonical decomposition of M follows since both sides of (3.2) are random measures coinciding for sets in \mathcal{H} . \square

Using exactly the same arguments as in [JS03], Chapter II, Proposition 2.9, the characteristics of M can be decomposed further:

Proposition 3.5 There exist $\tilde{\mathcal{P}}$ -measurable functions $b(\omega, t, x)$, $c(\omega, t, x)$, a transition kernel $K(\omega, t, x, dy)$ from $(\tilde{\Omega}, \tilde{\mathcal{P}})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and a positive predictable strict random measure $A(\omega, dt, dx)$ such that for all $\omega \in \Omega$,

$$\begin{aligned} B(\omega, dt, dx) &= b(\omega, t, x) A(\omega, dt, dx), \quad C(\omega, dt, dx) = c(\omega, t, x) A(\omega, dt, dx), \\ \nu(\omega, dt, dx, dy) &= K(\omega, t, x, dy) A(\omega, dt, dx). \end{aligned}$$

The next proposition determines the characteristics of the null-spatial random measure $H \cdot M$ for some given H :

Proposition 3.6 Let $H \in \tilde{\mathcal{P}}$ satisfy (2.7) and M be a spatially independently scattered random measure with different times of discontinuity. Then the null-spatial random measure $H \cdot M$ has characteristics $(B^{H \cdot M}, C^{H \cdot M}, \nu^{H \cdot M})$ given by

$$\begin{aligned} B^{H \cdot M}(dt) &= (H \cdot B)(dt) + \int_{E \times \mathbb{R}} [\tau(H(t, x)y) - H(t, x)\tau(y)] \nu(dt, dx, dy), \\ C^{H \cdot M}(dt) &= \int_E H^2(t, x) C(dt, dx), \quad W(t, y) * \nu^{H \cdot M} = W(t, H(t, x)y) * \nu, \end{aligned}$$

where W is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$ -measurable function such that $W(t, y) * \nu^{H \cdot M}$ exists.

Proof. It suffices to consider $H \in L^{1,0}(M)$ because for general $H \in L^0(M)$ we can find a sequence $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{P}$ with $A_n \uparrow \Omega$ and $H1_{A_n} \in L^{1,0}(M)$ for all $n \in \mathbb{N}$. As before, we then have $H1_{A_n} \cdot M \rightarrow H \cdot M$ in \mathcal{SM} , which implies the convergence of the characteristics by Lemma 3.4(1).

So let us turn to $H \in L^{1,0}(M)$. We first take some $D \in \mathcal{P} \otimes \mathcal{B}_0(\mathbb{R})$ and claim that

$$1_D(s, y) * \mu^{H \cdot M} = 1_D(s, H(s, x)y) * \mu. \quad (3.4)$$

This identity immediately extends to finite linear combinations of such indicators and thus, by the DCT 2.5, also to all functions $W(\omega, t, y)$ for which $W * \mu^{H \cdot M}$ exists. By the definition of the predictable compensator, this statement also passes to the case where μ is replaced by ν .

In order to prove (3.4), first observe that the jump process of the semimartingale $H \cdot M$ up to infinity is given by $\Delta(H \cdot M)_t = (H \cdot M)(\Omega \times \{t\} \times E)$. Furthermore, we can assume

that D does not contain any points in $\bar{\Omega} \times \{0\}$. Hence, in the case where $H = 1_A$ with $A \in \tilde{\mathcal{P}}_M$, we have for all $t \in \mathbb{R}$

$$1_D(s, y) * \mu_t^{H \cdot M} = 1_D(s, y) * \mu_t^{1_A \cdot M} = 1_D(s, y) 1_A(s, x) * \mu_t = 1_D(s, 1_A(s, x)y) * \mu_t.$$

Now a similar calculation yields that (3.4) remains true for all functions $H \in \mathcal{S}_M$. Finally, let $H \in L^{1,0}(M)$. By decomposing $H = H^+ - H^-$ into its positive and negative part, we may assume that $H \geq 0$ and choose a sequence $(H_n)_{n \in \mathbb{N}}$ of simple functions with $H_n \uparrow H$ as $n \rightarrow \infty$. As we have already seen in the proof of Theorem 3.3, we have $1_D(s, y) * \mu^{H_n \cdot M} \rightarrow 1_D(s, y) * \mu^{H \cdot M}$ in \mathcal{SM} . On the other hand, if D is of the form $R \times (a, b]$ with $R \in \mathcal{P}$ and $(a, b] \subseteq (0, \infty)$ or of the form $R \times [a, b)$ with $[a, b) \subseteq (-\infty, 0)$, then $1_D(\omega, s, H_n(\omega, s, x)y) \rightarrow 1_D(\omega, s, H(\omega, s, x)y)$ as $n \rightarrow \infty$ for every $(\omega, s, x, y) \in \tilde{\Omega} \times \mathbb{R}$, which shows that (3.4) holds up to indistinguishability. For general D , use Dynkin's π - λ -lemma (cf. [Bil95], Theorem 3.2).

Now consider $C^{H \cdot M} = [H \cdot M^c]$. First let $H = \sum_{i=1}^r a_i 1_{A_i}$ be a finite linear combination of pairwise disjoint sets in $\tilde{\mathcal{P}}_M$. Then we obviously have $H^2 = \sum_{i=1}^r a_i^2 1_{A_i}$ and thus

$$\int_{(-\infty, t] \times E} H^2(s, x) C(ds, dx) = \sum_{i=1}^r a_i^2 [1_{A_i} \cdot M^c]_t = \left[\sum_{i=1}^r a_i 1_{A_i} \cdot M^c \right]_t = C_t^{H \cdot M}, \quad t \in \mathbb{R},$$

where the second equality is due to the spatial independence of M . Now let $H \in L^{1,0}(M)$ and $(H_n)_{n \in \mathbb{N}} \subseteq \mathcal{S}_M$ converge pointwise to H with $|H_n| \leq |H|$ for all $n \in \mathbb{N}$. Then $[H_n \cdot M^c]$ and $\int H_n^2(s, x) 1_{(-\infty, t] \times E}(s, x) C(ds, dx)$ converge in \mathcal{SM} to the corresponding expressions with H_n replaced by H , which shows their identity by what we have shown so far.

Finally, we compute $B^{H \cdot M}$. The results up to now yield that for all $t \in \mathbb{R}$,

$$\begin{aligned} (H \cdot M)_t - (y - \tau(y)) * \mu_t^{H \cdot M} &= (H \cdot B)_t + (H \cdot M^c)_t + H(s, x)(y - \tau(y)) * \mu_t + \\ &\quad + H(s, x)\tau(y) * (\mu - \nu)_t - [H(s, x)y - \tau(H(s, x)y)] * \mu_t. \end{aligned}$$

By definition, $B^{H \cdot M}$ is the finite variation part in the canonical decomposition of this special semimartingale, which exactly equals $H \cdot B + [\tau(H(t, x)y) - H(t, x)\tau(y)] * \nu$. \square

4 An integrability criterion

The next theorem gives a complete characterization of $L^0(M)$ in terms of the characteristic triplet of M . In the null-spatial case one can consult [JS03], Chapter III, §6d, [CS05] and [BOGP13]. Moreover, for random measures M which are spatially independently scattered with different times of discontinuity, the spaces $L^0(M)$ and $L^{1,0}(M)$ are equal. Our theorem and its proof are a generalization of Theorem 4.5 in [BOGP13]:

Theorem 4.1 (Characterization of $L^0(M)$)

Let M be a spatially independently scattered random measure with different times of discontinuity whose characteristics w.r.t. some truncation function τ factorize as in Proposition 3.5. Then we have $L^0(M) = L^{1,0}(M)$ and for some $H \in \tilde{\mathcal{P}}$ we have $H \in L^0(M)$ if and only if each of the following conditions is satisfied a.s.:

- (1) $\int_{\mathbb{R} \times E} \left| H(t, x)b(t, x) + \int_{\mathbb{R}} [\tau(H(t, x)y) - H(t, x)\tau(y)] K(t, x, dy) \right| A(dt, dx) < \infty,$
- (2) $\int_{\mathbb{R} \times E} H^2(t, x)c(t, x) A(dt, dx) < \infty,$
- (3) $\int_{\mathbb{R} \times E} \int_{\mathbb{R}} (1 \wedge (H(t, x)y)^2) K(t, x, dy) A(dt, dx) < \infty.$

For the proof we first introduce some abbreviations: For $t \in \mathbb{R}$, $x \in E$ and $a \in \mathbb{R}$ define

- $U(t, x, a) := \left| ab(t, x) + \int_{\mathbb{R}} (\tau(ay) - a\tau(y)) K(t, x, dy) \right|,$
- $\tilde{U}(t, x, a) := \sup_{-1 \leq c \leq 1} U(t, x, ca).$

The following lemma is a straightforward extension of [RR89], Lemma 2.8, and will be needed in the proof of Theorem 4.1:

Lemma 4.2 There exists a constant $\kappa > 0$ such that

$$\tilde{U}(t, x, a) \leq U(t, x, a) + \kappa \int_{\mathbb{R}} (1 \wedge (ay)^2) K(t, x, dy).$$

Proof. Fix some $0 < \epsilon < 1$ and $N > 0$ such that $\tau(y) = y$ for $|y| \leq \epsilon$ and $|\tau(y)| \leq N$ for all $y \in \mathbb{R}$. If $|c| \leq 1$, $t \in \mathbb{R}$, $x \in E$ and $a \in \mathbb{R}$, we have

$$\begin{aligned} U(t, x, ca) &= \left| cab(t, x) + \int_{\mathbb{R}} (\tau(cay) - ca\tau(y)) K(t, x, dy) \right| \\ &= \left| cab(t, x) + c \int_{\mathbb{R}} (\tau(ay) - a\tau(y)) K(t, x, dy) + \int_{\mathbb{R}} (\tau(cay) - c\tau(ay)) K(t, x, dy) \right| \\ &\leq U(t, x, a) + |R(c, t, x, a)|, \end{aligned}$$

where $R(c, t, x, a)$ denotes the last summand in the second line. Since $|\tau(cay) - c\tau(ay)|$ is 0 for $|ay| \leq \epsilon$ and always less or equal to $(1 + |c|)N \leq 2N$, Chebyshev's inequality implies

$$\begin{aligned} |R(c, t, x, a)| &\leq 2N \int_{\{|ay| \geq \epsilon\}} K(t, x, dy) = 2N K(t, x, \{y \in \mathbb{R} : |ay| \wedge 1 \geq \epsilon\}) \\ &\leq \frac{2N}{\epsilon^2} \int_{\mathbb{R}} ((ay)^2 \wedge 1) K(t, x, dy). \end{aligned}$$

Choosing $\kappa := 2N\epsilon^{-2}$ completes the proof. \square

Proof of Theorem 4.1. We first prove that $H \in L^0(M)$ implies (1)-(3). Since $H \cdot M$ is a semimartingale up to infinity, $B^{H \cdot M}(\mathbb{R})$ and $C^{H \cdot M}(\mathbb{R})$ exist. By Proposition 3.6, this exactly gives the first two conditions. For the last condition, observe that it suffices to

show that $(1 \wedge y^2) * \nu_\infty^{H \cdot M} < \infty$ a.s., which is equivalent to $(1 \wedge y^2) * \mu_\infty^{H \cdot M} < \infty$ a.s. Obviously, $1_{\{|y| \geq 1\}} * \mu_\infty^{H \cdot M} < \infty$. In addition, we obtain

$$y^2 1_{\{|y| \leq 1\}} * \mu_\infty^{H \cdot M} = \sum_{t \in \mathbb{R}} (\Delta(H \cdot M)_t)^2 1_{\{|\Delta(H \cdot M)_t| \leq 1\}} \leq [H \cdot M, H \cdot M]_\infty < \infty,$$

which completes the first part of the proof.

Since $L^{1,0}(M) \subseteq L^0(M)$ is clear, the proof is complete if we can show that the three conditions in Theorem 4.1 imply $H \in L^{1,0}(M)$. To this end, define the collection

$$\mathcal{D} := \{G \in \tilde{\mathcal{P}} : |G| \leq 1, GH \in L^{1,0}(M)\}.$$

By Theorem 2.4 and the fact that for predictable functions H the Daniell mean equals

$$\|H\|_{M,0}^D = \sup_{S \in \mathcal{S}_M, |S| \leq |H|} \left\| \int S \, dM \right\|_0 = \sup_{G \in \tilde{\mathcal{P}}, |G| \leq 1, GH \in L^{1,0}(M)} \left\| \int GH \, dM \right\|_0,$$

we have to show that the set $\{\int GH \, dM : G \in \mathcal{D}\}$ is bounded in L^0 (which means bounded in probability) whenever H satisfies the three given conditions. By Proposition 3.6,

$$\int GH \, dM = \int GH \, dM^c + \tau(GHy) * (\mu - \nu)_\infty + (GHy - \tau(GHy)) * \mu_\infty + B^{GH \cdot M}(\mathbb{R}).$$

We consider each part of this formula separately and show that each of the sets

$$\{B^{GH \cdot M}(\mathbb{R}) : G \in \mathcal{D}\}, \quad (4.1)$$

$$\{\int GH \, dM^c : G \in \mathcal{D}\}, \quad (4.2)$$

$$\{\tau(GHy) * (\mu - \nu)_\infty : G \in \mathcal{D}\}, \quad (4.3)$$

$$\{(GHy - \tau(GHy)) * \mu_\infty : G \in \mathcal{D}\} \quad (4.4)$$

is bounded in probability.

If $G \in \mathcal{D}$ and $\kappa > 0$ denotes the constant in Lemma 4.2, we can use the first and the third condition to obtain

$$\begin{aligned} & \int_{\mathbb{R} \times E} U(t, x, G(t, x)H(t, x)) A(dt, dx) \leq \int_{\mathbb{R} \times E} \tilde{U}(t, x, G(t, x)H(t, x)) A(dt, dx) \\ & \leq \int_{\mathbb{R} \times E} \tilde{U}(t, x, H(t, x)) A(dt, dx) \\ & \leq \int_{\mathbb{R} \times E} U(t, x, H(t, x)) A(dt, dx) + \kappa \int_{\mathbb{R} \times E} \int_{\mathbb{R}} (1 \wedge (H(t, x)y)) K(t, x, dy) A(dt, dx) < \infty \end{aligned}$$

a.s., which shows that (4.1) is bounded in probability.

Next consider (4.2) and fix some $G \in \mathcal{D}$ for a moment. For all $\epsilon, \eta > 0$, Lengart's

inequality (see [JS03], Lemma 3.30a, for instance) shows that

$$\begin{aligned}
P \left[\left| \int GH \, dM^c \right| \geq \epsilon \right] &\leq P \left[\sup_{t \in \mathbb{R}} |(GH \cdot M)^c(\bar{\Omega}_t)| \geq \epsilon \right] = P \left[\sup_{t \in \mathbb{R}} |(GH \cdot M)^c(\bar{\Omega}_t)|^2 \geq \epsilon^2 \right] \\
&\leq \frac{\eta}{\epsilon^2} + P \left[[(GH \cdot M)^c]_\infty \geq \eta \right] = \frac{\eta}{\epsilon^2} + P[C^{GH \cdot M}(\mathbb{R}) \geq \eta] \\
&= \frac{\eta}{\epsilon^2} + P \left[\int_{\mathbb{R} \times E} G^2(t, x) H^2(t, x) c(t, x) A(dt, dx) \geq \eta \right] \\
&\leq \frac{\eta}{\epsilon^2} + P \left[\int_{\mathbb{R} \times E} H^2(t, x) c(t, x) A(dt, dx) \geq \eta \right].
\end{aligned}$$

By the second condition, we can, given some fix $\delta > 0$, first choose $\eta > 0$ so large that the last probability is less than $\frac{\delta}{2}$ and then choose $\epsilon > 0$ large enough such that also $\frac{\eta}{\epsilon^2} < \frac{\delta}{2}$. This shows that the set (4.2) is bounded in probability.

For (4.3), we use the abbreviation $W(t, x, y) = \tau(G(t, x)H(t, x)y)$. Using Lengart's inequality, we obtain for every $\epsilon, \eta > 0$:

$$\begin{aligned}
P[|W * (\mu - \nu)_\infty| \geq \epsilon] &\leq P \left[\sup_{t \in \mathbb{R}} |W * (\mu - \nu)_t|^2 \geq \epsilon^2 \right] \\
&\leq \frac{\eta}{\epsilon^2} + P[|W * (\mu - \nu)|_\infty \geq \eta].
\end{aligned} \tag{4.5}$$

Now recall that $\int_{\mathbb{R}} \tau(y) \nu^{GH \cdot M}(\{t\} \times dy) = \Delta B_t^{GH \cdot M}$ (see [JS03], Chapter II, Proposition 2.9) and that by the definition of the semimartingale characteristics we can find some $\epsilon > 0$ such that a jump of $B^{GH \cdot M}$ at time $t \in \mathbb{R}$ with $|\Delta B_t^{GH \cdot M}| \in (0, \epsilon)$ results in a jump of $GH \cdot M$ at the same time and of the same size. As a result, if we choose $r, q > 0$ such that $|\tau(y)| \leq r(1 \wedge |y|)$ for all $y \in \mathbb{R}$ and $|\tau(y)| \leq q(1 \wedge y^2)$ for all $|y| \geq \epsilon$, we can use Proposition 3.6 to obtain

$$\begin{aligned}
[W * (\mu - \nu)]_\infty &= [\tau(y) * (\mu^{GH \cdot M} - \nu^{GH \cdot M})]_\infty \\
&= \sum_{t \in \mathbb{R}} \left[\tau(\Delta(GH \cdot M)_t) - \int_{\mathbb{R}} \tau(y) \nu^{GH \cdot M}(\{t\} \times dy) \right]^2 \\
&= \sum_{t \in \mathbb{R}} [\tau(\Delta(GH \cdot M)_t) - \Delta B_t^{GH \cdot M}]^2 \\
&\leq 2 \sum_{t \in \mathbb{R}} [\tau(\Delta(GH \cdot M)_t)]^2 + 2 \sum_{t \in \mathbb{R}} [\Delta B_t^{GH \cdot M} 1_{\{|\Delta B_t^{GH \cdot M}| \geq \epsilon\}}]^2 \\
&\leq 2r^2 \sum_{t \in \mathbb{R}} (1 \wedge (\Delta(GH \cdot M)_t)^2) + 2 \sum_{t \in \mathbb{R}} \left[\int_{\{|y| \geq \epsilon\}} \tau(y) \nu^{GH \cdot M}(\{t\} \times dy) \right]^2 \\
&\leq 2r^2(1 \wedge (GHy)^2) * \mu_\infty + 2q^2 \left(\sum_{t \in \mathbb{R}} \int_{\mathbb{R}} (1 \wedge y^2) \nu^{GH \cdot M}(\{t\} \times dy) \right)^2 \\
&\leq 2r^2(1 \wedge (Hy)^2) * \mu_\infty + 2q^2((1 \wedge (Hy)^2) * \nu_\infty)^2,
\end{aligned}$$

which is a.s. finite because of the third condition together with Lemma 3.4(2). This yields the boundedness of (4.3).

Eventually, choose $r, \epsilon > 0$ such that $f(y) := r|y|1_{\{|y|>\epsilon\}}$ satisfies $|y - \tau(y)| \leq f(y)$ for all $y \in \mathbb{R}$. Obviously, f is symmetric and increasing on \mathbb{R}_+ so that

$$|(GHy - \tau(GHy)) * \mu_\infty| \leq f(GHy) * \mu_\infty \leq f(Hy) * \mu_\infty.$$

Now the third condition and Lemma 3.4(2) imply that

$$\sum_{t \in \mathbb{R}} (1 \wedge \epsilon^2) 1_{\{|\Delta(H \cdot M)_t| > \epsilon\}} \leq (1 \wedge y^2) * \mu_\infty^{H \cdot M} = (1 \wedge (H(t, x)y)^2) * \mu_\infty < \infty$$

a.s. such that $\{|\Delta(H \cdot M)_t| > \epsilon\}$ only happens for finitely many time points. Hence

$$f(Hy) * \mu_\infty = f(y) * \mu_\infty^{H \cdot M} = r \sum_{t \in \mathbb{R}} |\Delta(H \cdot M)_t| 1_{\{|\Delta(H \cdot M)_t| > \epsilon\}} < \infty$$

a.s., which implies that the set in (4.4) is also bounded in probability. This finishes the proof of Theorem 4.1. \square

Some remarks are in order:

Remark 4.3 In applications one often encounters random measures M with summable jumps, which means that each of the semimartingales $(M(\tilde{\Omega}_t \cap \tilde{O}_k))_{t \in \mathbb{R}}$, $k \in \mathbb{N}$, has summable jumps over finite intervals. Then it is often convenient to take the truncation function $\tau = 0$, which is not a proper truncation function as defined on p. 7. In this case one has to change the third condition of Theorem 4.1 into

$$\int_{\mathbb{R} \times E} \int_{\mathbb{R}} (1 \wedge |H(t, x)y|) K(t, x, dy) A(dt, dx) < \infty. \quad (4.6)$$

Together with (1) and (2) of Theorem 4.1, (4.6) is again equivalent to the integrability of H : First note that we can choose $\kappa = 0$ in Lemma 4.2(2) since τ is identical 0 and therefore $\tilde{U} = U$. So the calculations done for (4.1) remain valid. Moreover, (4.2) does not depend on τ and the boundedness of (4.3) becomes trivial. For (4.4) observe that

$$|GHy| * \mu_\infty \leq |Hy| * \mu_\infty = |y| * \mu_\infty^{H \cdot M} = |y| 1_{\{|y| \leq 1\}} * \mu_\infty^{H \cdot M} + |y| 1_{\{|y| > 1\}} * \mu_\infty^{H \cdot M}. \quad (4.7)$$

Now (4.6) implies by Lemma 3.4(2) that a.s.,

$$|y| 1_{\{|y| \leq 1\}} * \mu_\infty^{H \cdot M} + 1_{\{|y| > 1\}} * \mu_\infty^{H \cdot M} = \int_{\mathbb{R} \times E} \int_{\mathbb{R}} (1 \wedge |H(t, x)y|) K(t, x, dy) A(dt, dx) < \infty.$$

As a result, on the right-hand side of (4.7), the first summand converges a.s. and the second one is in fact just a finite sum a.s.

On the other hand, the assumption that M has summable jumps implies that whenever $H \in L^0(M)$, $H \cdot (M - M^c)$ is a semimartingale up to infinity with finite variation on \mathbb{R} . As a result we have

$$\int_{\mathbb{R}} 1 \wedge |y| \mu^{H \cdot M}(dy) \leq |H \cdot (M - M^c)|_\infty < \infty \quad \text{a.s.,}$$

which implies (4.6) by Lemma 4.2(2) and Proposition 3.6.

Remark 4.4 Assume the case where the integrand H is independent of M . Then it suffices to check the three conditions in Theorem 4.1 for the realisations $H(\omega)$ of H . If they hold for a.e. $H(\omega)$, then H is integrable w.r.t. M . In fact, if $P|(H = \bar{H})$ denotes the regular conditional probability given $H = \bar{H}$, where \bar{H} is any function $\mathbb{R} \times E \rightarrow \mathbb{R}$, and P_H denotes the image measure of P under H , then M still has the same characteristics under $P|(H = \bar{H})$ by independence and

$$\begin{aligned} P[B^{H \cdot M}(\mathbb{R}) < \infty] &= \mathbb{E} \left[P[B^{H \cdot M}(\mathbb{R}) < \infty | H] \right] \\ &= \int_{\mathbb{R} \times E} P[B^{H \cdot M}(\mathbb{R}) < \infty | H = \bar{H}] P_H(d\bar{H}) = \int_{\mathbb{R} \times E} P[B^{\bar{H} \cdot M}(\mathbb{R}) < \infty] P_H(d\bar{H}) = 1. \end{aligned}$$

Since the other two conditions of Theorem 4.1 can be treated analogously, the assertion is proved. Let us also remark that this result can already be derived from [RR89], Theorem 2.7. So the main contribution of Theorem 4.1 actually lies in the case where H and M are *not* independent.

Remark 4.5 The assumption that M is spatially independently scattered in Theorem 3.3 and Theorem 4.1 was only needed for C and the second condition, respectively. If one does not assume the spatial independence of M , C is not a measure anymore since then even finite additivity fails. However, it is a bimeasure in the sense that for fixed $A \in \tilde{\mathcal{P}}_M$ both $C(\cdot; A)$ and $C(A; \cdot)$ given by

$$C(B; A) = C(A; B) := [1_A \cdot M^c, 1_B \cdot M^c]_\infty, \quad B \in \tilde{\mathcal{P}}_M,$$

are predictable strict random measures. It is well-known that bimeasures on \mathbb{R} cannot be extended to measures on the product σ -field in general and that integration theory w.r.t. bimeasures differs from integration theory w.r.t. measures. We briefly recall how integrals w.r.t. bimeasures are defined and follow [CR83]: Given two measurable spaces $(\Omega_i, \mathcal{F}_i)$, $i = 1, 2$, and a bimeasure $\beta: \mathcal{F}_1 \times \mathcal{F}_2 \rightarrow \mathbb{R}$, we call a pair (f_1, f_2) of \mathcal{F}_i -measurable functions f_i , $i = 1, 2$, **strictly β -integrable** if

- (1) f_1 (resp. f_2) is integrable w.r.t. $\beta(\cdot; B)$ for all $B \in \mathcal{F}_2$ (resp. $\beta(A; \cdot)$ for all $A \in \mathcal{F}_1$),
- (2) f_2 is integrable w.r.t. the measure $B \mapsto \int_{\Omega_1} f_1(\omega_1) \beta(d\omega_1; B)$ and f_1 is integrable w.r.t. the measure $A \mapsto \int_{\Omega_2} f_2(\omega_2) \beta(A; d\omega_2)$,
- (3) for all $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$, the following integrals are equal:

$$\int_A f_1(\omega_1) \left(\int_B f_2(\omega_2) \beta(d\omega_1; d\omega_2) \right) = \int_B f_2(\omega_2) \left(\int_A f_1(\omega_1) \beta(d\omega_1; d\omega_2) \right).$$

The **strict β -integral** of $(f_1; f_2)$ on $(A; B)$, denoted by $\int_{(A; B)} (f_1; f_2) d\beta$, is then defined as this common value. Our first aim is now to show the following:

Lemma 4.6 If $H \in L^{1,0}(M^c)$, then $(H; H)$ is strictly C -integrable and

$$[H \cdot M^c]_t = \int_{(\bar{\Omega}_t; \bar{\Omega}_t)} (H; H) dC, \quad t \in \mathbb{R}.$$

Moreover, if $H \in L^0(M^c)$ and K denotes the strictly positive process in (2.7), $(HK; HK)$ is strictly C -integrable and K^{-2} is integrable w.r.t. $C^{HK \cdot M^c}$ with

$$[H \cdot M^c]_t = K^{-2} \cdot \int_{(\tilde{\Omega}_t; \tilde{\Omega}_t)} (HK; HK) dC, \quad t \in \mathbb{R}. \quad (4.8)$$

Proof. Since the second part follows immediately from the first part, it suffices to consider $H \in L^{1,0}(M^c)$. To this end, let $(H_n)_{n \in \mathbb{N}}$ be a sequence of simple integrands with $|H_n| \leq |H|$ for all $n \in \mathbb{N}$ and $H_n \rightarrow H$ pointwise. Since for simple integrands the claim follows directly from the definition of C and the bimeasure integral, we can use the DCT 2.5 and Lemma 3.4(1) on the one hand and the DCT for bimeasure integrals (see [CR83], Corollary 2.9) on the other hand to obtain the result. The only property to show is that $(H; H)$ is strictly C -integrable, which means by the symmetry of C the following two points: first, that H is integrable w.r.t. the measure $A \mapsto C(A; B) = [1_A \cdot M^c, 1_B \cdot M^c]_\infty$ for all $B \in \tilde{\mathcal{P}}_{M^c}$, and second, that H is integrable w.r.t. the measure $A \mapsto \int H(t, x) dC(A; dt, dx)$, which equals $[1_A \cdot M^c, H \cdot M^c]_\infty$ by the preceding arguments. So let G be 1_B or H . From [Leb95a], Theorem 2 and its Corollary, we know that there exists a probability measure Q equivalent to P such that M^c is an $L^2(Q)$ -valued σ -finite measure with $G, H \in L^{1,2}(M^c; Q)$. Since the bounded sets in $L^0(P)$ are exactly the bounded sets in $L^0(Q)$, convergence in $\|\cdot\|_{M^c,0;P}^D$ is equivalent to convergence in $\|\cdot\|_{M^c,0;Q}^D$. Similarly, stochastic integrals and predictable quadratic covariation remain unchanged under Q (see [Bic02], Proposition 3.6.20, and [JS03], Theorem 4.47a). Consequently, if we write $\gamma(A) := [1_A \cdot M^c, G \cdot M^c]_\infty$ for $A \in \tilde{\mathcal{P}}_{M^c}$, it suffices to show that

$$\sup_{S \in \mathcal{S}_{M^c}, |S| \leq |rH|} \left\| \int S d\gamma \right\|_{L^0(Q)} = \sup_{S \in \mathcal{S}_{M^c}, |S| \leq |rH|} \|[S \cdot M^c, G \cdot M^c]_\infty\|_{L^0(Q)} \rightarrow 0 \quad \text{as } r \rightarrow 0.$$

Indeed, using Fefferman's inequality (cf. [Pro05], Chapter 4, Theorem 52), we can find a constant $R > 0$, which only depends on G , such that

$$\begin{aligned} \sup_{S \in \mathcal{S}_{M^c}, |S| \leq |rH|} \|[S \cdot M^c, G \cdot M^c]_\infty\|_{L^0(Q)} &\leq R \sup_{S \in \mathcal{S}_{M^c}, |S| \leq |rH|} \mathbb{E}_Q[[S \cdot M^c]_\infty]^{1/2} \\ &= R \sup_{S \in \mathcal{S}_{M^c}, |S| \leq |rH|} \|S \cdot M_\infty^c\|_{L^2(Q)} = R \|rH\|_{M^c,2;Q}^D \rightarrow 0 \end{aligned}$$

as $r \rightarrow 0$, which finishes the proof. \square

As a result, in the case where M is not assumed to be spatially independently scattered, we obtain the following version of Theorem 4.1 (we only state the condition for M^c , the other parts remain the same):

Theorem 4.7 Let H be a predictable function satisfying (2.7) with some $K > 0$. Then $H \in L^0(M^c)$ if and only if (4.8) is a.s. finite.

Proof. The necessity of (4.8) is clear from the preceding lemma. For sufficiency let us consider the set (4.2) with $\mathcal{D} := \{G \in \mathcal{P}: |G| \leq 1, GH \in L^{1,0}(M^c)\}$. Then again by

Lenglart's inequality, we have

$$\begin{aligned} P \left[\left| \int GH \, dM^c \right| \geq \epsilon \right] &\leq P \left[\sup_{t \in \mathbb{R}} |(GH \cdot M^c)(\bar{\Omega}_t)| \geq \epsilon \right] = P \left[\sup_{t \in \mathbb{R}} |(GH \cdot M^c)(\bar{\Omega}_t)|^2 \geq \epsilon^2 \right] \\ &\leq \frac{\eta}{\epsilon^2} + P[[GH \cdot M^c]_\infty \geq \eta] = \frac{\eta}{\epsilon^2} + P[G^2 K^{-2} \cdot [KH \cdot M^c]_\infty \geq \eta] \\ &\leq \frac{\eta}{\epsilon^2} + P[K^{-2} \cdot [KH \cdot M^c]_\infty \geq \eta]. \end{aligned}$$

By (4.8) we can first choose $\eta > 0$ and then $\epsilon > 0$, both independently of $G \in \mathcal{D}$, to make the quantity on the left-hand side arbitrarily small. By (2.9) the assertion follows. \square

Let us also remark that the inclusion $L^{1,0}(M) \subseteq L^0(M)$ may be strict now.

Remark 4.8 In applications, one is often interested in a multi-dimensional random measure $M = (M^1, \dots, M^d)^\top$ as integrator, where M^i is an L^0 -valued σ -finite random measure on E for all $i = 1, \dots, d$, and a matrix-valued function H as integrand. By considering the rows of H separately, we may assume $H = (H^1, \dots, H^d)$. From the point of view of stochastic integration, the theory is the same as in the one-dimensional case as one can see as follows: for each $i = 1, \dots, d$, let $E^{(i)}$ consist of all elements in E labeled with the number i such that we have $x^{(i)} \in E^{(i)}$ if and only if $x \in E$. Denote $\hat{E} = \bigcup_{i=1}^d E^{(i)}$ and define a new random measure on \hat{E} by

$$\hat{M} \left(S \times \bigcup_{i=1}^d U_i^{(i)} \right) := \sum_{i=1}^d M^i(S \times U_i), \quad S \in \mathcal{P}, U_i^{(i)} \in \mathcal{E}^{(i)}.$$

Further define a new function $\hat{H}: \Omega \times \mathbb{R} \times \hat{E} \rightarrow \mathbb{R}$ by

$$\hat{H}(\omega, t, x^{(i)}) := H^i(\omega, t, x), \quad \omega \in \Omega, t \in \mathbb{R}, x \in E, i = 1, \dots, d.$$

Naturally, we would make the following definition: the function H is in $L^{1,p}(M)$ if and only if \hat{H} is in $L^{1,p}(\hat{M})$ and we set

$$\int H \, dM := \int \hat{H} \, d\hat{M}.$$

Furthermore, if there exists a predictable process $K > 0$ with $HK \in L^{1,p}(M)$, we can define the one-dimensional null-spatial random measure $H \cdot M := \hat{H} \cdot \hat{M}$ and if it is a finite L^p -valued random measure, we write $H \in L^p(M)$ and set

$$\int H \, dM := \hat{H} \cdot \hat{M}(\bar{\Omega}).$$

The difference comes up when we want to apply the canonical decomposition as in Theorem 3.3 or the integrability conditions in Theorem 4.1: in the multi-dimensional case it is no longer reasonable to assume that the random measures M^i have different times of discontinuity or that they are independent. As a result, we have to define the characteristic triplet (B, C, ν) of M also in a multi-dimensional way, similar to the approaches in [JS03], Chapter 6, or [BOGP13]. Using their methods, we obtain the following theorem as a straightforward extension of the results in the last Section. We omit the proof:

Theorem 4.9 Let $M = (M^1, \dots, M^d)^\top$ be a d -dimensional spatially independently scattered random measure with different times of discontinuity, which means that for all disjoint sets $U_1, U_2 \in \mathcal{E}_M$ and $k \in \mathbb{N}$ the vector-valued semimartingales $M((O_k \cap \bar{\Omega}) \times U_i)$, $i = 1, 2$, are independent and never jump at the same time. For $A \in \tilde{\mathcal{P}}_M$ denote the d -dimensional semimartingale $(1_A \cdot M^1, \dots, 1_A \cdot M^d)^\top$ by $1_A \cdot M$. Given a truncation function $\tau: \mathbb{R}^d \rightarrow \mathbb{R}^d$, let us set

$$B := \mathfrak{B}(1_A \cdot M)_\infty, \quad M^c := (M^{1,c}, \dots, M^{d,c})^\top, \quad C^{ij}(A) := [M^i(A), M^j(A)]_\infty^c, \\ \mu(A, V) := \mu^{1_A \cdot M}(\mathbb{R}, V), \quad \nu(A, V) := \nu^{1_A \cdot M}(\mathbb{R}, V), \quad A \in \tilde{\mathcal{P}}_M, V \in \mathcal{B}_0(\mathbb{R}^d).$$

Then the following holds:

- (1) The characteristics (B, C, ν) of M are unique and can be chosen as predictable strict random measures. The canonical decomposition of M is as follows:

$$M(A) = B(A) + M^c(A) + \int_{\mathbb{R} \times E \times \mathbb{R}^d} 1_A(t, x)(y - \tau(y)) \mu(dt, dx, dy) + \\ + \int_{\mathbb{R} \times E \times \mathbb{R}^d} 1_A(t, x)\tau(y) (\mu - \nu)(dt, dx, dy), \quad A \in \tilde{\mathcal{P}}_M.$$

- (2) Analogously to Proposition 3.5, there exists a disintegration (b, c, K) of the characteristic triplet w.r.t. a positive predictable strict random measure A . Here, b and c are predictable functions with the former one taking values in \mathbb{R}^d and the latter one in the set of positive semidefinite $(d \times d)$ -matrices, and K is a transition kernel from $(\tilde{\Omega}, \tilde{\mathcal{P}})$ to $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.
- (3) Let H be a d -dimensional predictable function such that there exists a predictable process $K > 0$ with $HK \in L^{1,0}(M)$. Then the characteristics of $H \cdot M$ are given by

$$B^{H \cdot M}(dt) = (H \cdot B)(dt) + \int_{E \times \mathbb{R}^d} [\tau(H(t, x)y) - H(t, x)\tau(y)] \nu(dt, dx, dy), \\ C^{H \cdot M}(dt) = \int_E H(t, x)c(t, x)H^\top(t, x) A(dt, dx), \quad W(t, y) * \nu^{H \cdot M} = W(t, H(t, x)y) * \nu$$

for all $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$ -measurable functions W such that $W * \nu^{H \cdot M}$ exists.

- (4) We have $H \in L^0(M)$ if and only if each of the following conditions is satisfied a.s.:

$$(a) \int_{\mathbb{R} \times E} \left| H(t, x)b(t, x) + \int_{\mathbb{R}^d} [\tau(H(t, x)y) - H(t, x)\tau(y)] K(t, x, dy) \right| A(dt, dx) < \infty, \\ (b) \int_{\mathbb{R} \times E} H(t, x)c(t, x)H^\top(t, x) A(dt, dx) < \infty, \\ (c) \int_{\mathbb{R} \times E} \int_{\mathbb{R}^d} (1 \wedge (H(t, x)y)^2) K(t, x, dy) A(dt, dx) < \infty.$$

Note that we again no longer have $L^{1,0}(M) = L^0(M)$.

5 Fubini's theorem for random measures

Theorem 5.1 Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, P)$ be a stochastic basis satisfying the usual assumptions.

- (1) Let m be a null-spatial random measure and $\lambda(\omega, t, dx)$ be a predictable signed transition kernel from $(\tilde{\Omega}, \tilde{\mathcal{P}})$ to (E, \mathcal{E}) . Further assume that there is a sequence $(\tilde{O}_k)_{k \in \mathbb{N}} \subseteq \tilde{\mathcal{P}}$ such that $\tilde{O}_k \uparrow \tilde{\Omega}$ and $\int_{\mathbb{R}} \int_E 1_{\tilde{O}_k}(t, x) \lambda(t, dx) m(dt)$ exists for every $k \in \mathbb{N}$. Then

$$M(A) := \int_{\mathbb{R}} \int_E 1_A(t, x) \lambda(t, dx) m(dt), \quad A \in \tilde{\mathcal{P}}_M,$$

defines a σ -finite random measure and for all $H \in L^{1,0}(M)$ we have

$$\int_{\mathbb{R} \times E} H(t, x) M(dt, dx) = \int_{\mathbb{R}} \int_E H(t, x) \lambda(t, dx) m(dt). \quad (5.1)$$

- (2) Let λ be a σ -finite signed measure on (E, \mathcal{E}) and $m(x, dt)$ be a null-spatial L^1 -random measure for each $x \in E$ such that $x \mapsto m(x, S)$ is \mathcal{E} -measurable for all $S \in \mathcal{P}_{m(x, \cdot)}$. Further assume that there exists a sequence $(\tilde{O}_k)_{k \in \mathbb{N}} \subseteq \tilde{\mathcal{P}}$ increasing to $\tilde{\Omega}$ such that

$$\int_E \|1_{\tilde{O}_k}(\cdot, x)\|_{m(x, \cdot), 1}^D \lambda(dx) < \infty.$$

Then

$$M(A) := \int_E \int_{\mathbb{R}} 1_A(t, x) m(x, dt) \lambda(dx), \quad A \in \tilde{\mathcal{P}}_M,$$

defines a σ -finite L^1 -valued random measure. Moreover, if $H \in \tilde{\mathcal{P}}$ satisfies

$$\int_E \|H(\cdot, x)\|_{m(x, \cdot), 1}^D \lambda(dx) < \infty, \quad (5.2)$$

we have $H \in L^{1,1}(M)$ and

$$\int_{\mathbb{R} \times E} H(t, x) M(dt, dx) = \int_E \int_{\mathbb{R}} H(t, x) m(x, dt) \lambda(dx). \quad (5.3)$$

Note that the statement in this theorem includes the existence of the right-hand side of (5.1) and (5.3). Note that in (5.1), the inner integral $(\omega, t) \mapsto \int_E H(\omega, t, x) \lambda(\omega, t, dx)$ defines a predictable process; in (5.3), the inner integral is a random variable and the outer integral is defined pointwise in ω . The second part of the theorem is inspired by the articles [BL95] and [Leb95b], where one can also find a generalization to the case of L^0 -valued random measures using prelocalization.

Proof. In both cases, properties (3) and (4) of Definition 2.1 are clear.

- (1) W.l.o.g., we can assume that λ is a positive kernel and that $H \geq 0$. That M does define a random measure is an easy consequence of the monotone convergence theorem, applied to λ and m consecutively. The inner integral on the right-hand side of (5.1)

is always defined for each $(\omega, t) \in \bar{\Omega}$, possibly taking the value $+\infty$. So if we write $\bar{H}(\omega, t) := \int_E H(\omega, t, x) \lambda(\omega, t, dx)$, then

$$\begin{aligned} \|\bar{H}\|_{m,0}^D &= \sup_{\substack{|S| \leq 1, S \in \mathcal{P} \\ S\bar{H} \in L^{1,0}(m)}} \left\| \int S \bar{H} dm \right\|_0 = \sup_{\substack{|S| \leq 1, S \in \mathcal{P} \\ S\bar{H} \in L^{1,0}(m)}} \left\| \int_{\mathbb{R}} \int_E S_t H(t, x) \lambda(t, dx) m(dt) \right\|_0 \\ &\leq \sup_{\tilde{S} \in \mathcal{S}_M, |\tilde{S}| \leq H} \left\| \int_{\mathbb{R}} \int_E \tilde{S}(t, x) \lambda(t, dx) m(dt) \right\|_0 = \sup_{\tilde{S} \in \mathcal{S}_M, |\tilde{S}| \leq H} \left\| \int \tilde{S} dM \right\|_0 = \|H\|_{M,0}^D, \end{aligned}$$

which shows the existence of the double integral in (5.1) as soon as $H \in L^{1,0}(M)$. As to (5.1), it obviously holds for simple functions. For general $H \in L^{1,0}(M)$, recall that $\bar{H} \in L^{1,0}(m)$, so applying the dominated convergence theorem twice finishes the proof.

(2) Again we assume that λ is a positive measure. That M defines a random measure, will be shown in the following as a byproduct. Now suppose that H satisfies (5.2). Then a simple calculation (see [BL95], Lemma 2.4) shows that

$$\|H\|_{M,1}^D \leq \int_E \|H(\cdot, x)\|_{m(x,\cdot),1}^D \lambda(dx) < \infty,$$

which implies $H \in L^{1,1}(M)$. Next we show that, if the right-hand side of (5.3) also exists in L^1 , then the integrals coincide. For simple functions this holds by definition. Now let H be integrable on both sides of (5.3) and choose simple functions $(S_n)_{n \in \mathbb{N}}$ with $S_n \rightarrow H$ pointwise and $|S_n| \leq |H|$. Then $\int S_n dM \rightarrow \int H dM$ in L^1 by the DCT 2.5. Similarly, we have for each $x \in E$ that $\int S_n(t, x) m(x, dt) \rightarrow \int H(t, x) m(x, dt)$ in L^1 . Then, since $\mathbb{E}[\|\int S_n(t, x) m(x, dt)\|] \leq \|H(\cdot, x)\|_{m(x,\cdot),1}^D$, which is integrable w.r.t. λ by assumption,

$$\int_E \mathbb{E} \left[\left\| \int_{\mathbb{R}} S_n(t, x) m(x, dt) \right\| \right] \lambda(dx) \rightarrow \int_E \mathbb{E} \left[\left\| \int_{\mathbb{R}} H(t, x) m(x, dt) \right\| \right] \lambda(dx),$$

by the dominated convergence theorem w.r.t. λ , which is the desired result after interchanging expectation and integral. In particular, this includes the σ -additivity of M , which shows that M is a random measure. Finally, we show the existence of the right-hand side of (5.3). To this end, let

$$\bar{H}(\omega, x) := \begin{cases} \left(\int H(t, x) m(x, dt) \right) (\omega), & H(\cdot, x) \in L^{1,0}(m(x, \cdot)), \\ +\infty & H(\cdot, x) \notin L^{1,0}(m(x, \cdot)). \end{cases} \quad \omega \in \Omega, x \in E.$$

Then we have to show that $\int_E |\bar{H}(\omega, x)| \lambda(dx)$ or, equivalently, $\int \bar{H}^\pm(\omega, x) \lambda(dx)$ is finite a.s. Let \mathcal{A} (resp. \mathcal{A}^\pm) denote the collection of all $\mathcal{F} \otimes \mathcal{E}$ -measurable functions S such that $|S| \leq 1$ and $|S(\omega, \cdot) \bar{H}(\omega, \cdot)|$ (resp. $0 \leq S \leq 1$ and $S(\omega, \cdot) \bar{H}^\pm(\omega, \cdot)$) is λ -integrable for all $\omega \in \Omega$. Then the proof can be completed by observing that

$$\begin{aligned} \left\| r \int_E \bar{H}^\pm(x) \lambda(dx) \right\|_0 &= \sup_{S \in \mathcal{A}^\pm} \left\| r \int_E S(x) \bar{H}^\pm(x) \lambda(dx) \right\|_0 \leq \sup_{S \in \mathcal{A}} \left\| r \int_E S(x) \bar{H}(x) \lambda(dx) \right\|_0 \\ &= \sup_{S \in \mathcal{A}} \left\| r \int_E S(x) \int_{\mathbb{R}} H(t, x) m(x, dt) \lambda(dx) \right\|_0 \\ &\leq \sup_{\tilde{S} \in \mathcal{S}_M, |\tilde{S}| \leq |rH|} \left\| \int \tilde{S} dM \right\|_0 = \|rH\|_{M,0}^D \rightarrow 0 \quad \text{as } r \rightarrow 0. \end{aligned}$$

□

6 Examples

In this section we present various examples, which partly motivated our investigation.

6.1 Lévy bases

Lévy bases, which are called infinitely divisible independently scattered random measures in [RR89], have attracted interest in several applications in the last few years, see e.g. [BN01], [BNS04], [BNS11], [BNBV11], [BNV12] and [BCK13]. In fact, Lévy bases form a key building block in all the examples below, so let us recall their definition: assume S is a non-empty set and \mathcal{S} a ring on S closed under countable intersections such that there exists a sequence $(S_k)_{k \in \mathbb{N}}$ in \mathcal{S} increasing to S . Then a **Lévy basis** is a mapping $\Lambda: \mathcal{S} \rightarrow L^0$ with the following properties:

- $\Lambda(\emptyset) = 0$ a.s.
- If $(A_n)_{n \in \mathbb{N}}$ is a sequence of pairwise disjoint sets in \mathcal{S} , then $(\Lambda(A_n))_{n \in \mathbb{N}}$ are independent random variables and if additionally $\bigcup_{n=1}^{\infty} A_n \in \mathcal{S}$, we have

$$\Lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \Lambda(A_n) \quad \text{in } L^0.$$

- For all $A \in \mathcal{S}$, $\Lambda(A)$ has an infinitely divisible distribution.

Note that in [RR89] convergence in L^0 in the second condition is replaced by a.s. convergence, but both are equivalent by the Itô-Nisio theorem (see [KW92], Theorem 2.1.1).

As shown in [RR89], Proposition 2.1 and Lemma 2.3, Λ induces a characteristic triplet (B, C, ν) w.r.t. some truncation function τ via the Lévy-Khintchine formula:

$$\mathbb{E}[e^{iu\Lambda(A)}] = \exp\left(iuB(A) - \frac{u^2}{2}C(A) + \int_{\mathbb{R}} (e^{iu\tau(y)} - 1 - iu\tau(y)) \nu(A, dy)\right), \quad A \in \mathcal{S}, u \in \mathbb{R}.$$

Moreover, B (resp. C) can be extended to a σ -finite signed (resp. positive) measure on $\sigma(\mathcal{S})$, ν to a σ -finite measure on $\sigma(\mathcal{S}) \otimes \mathcal{B}(\mathbb{R})$ and there exists another σ -finite measure λ on $\sigma(\mathcal{S})$ such that B , C and ν have a representation

$$B(ds) = b(s) \lambda(ds), \quad C(ds) = c(s) \lambda(ds), \quad \nu(ds, dy) = \kappa(s, dy) \lambda(ds) \quad (6.1)$$

with $\sigma(\mathcal{S})$ -measurable functions b and c and a transition kernel κ from (S, \mathcal{S}) to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Using this decomposition, [RR89] built a stochastic integration theory for deterministic integrands $f: S \rightarrow \mathbb{R}$, which is not only quite similar to our approach in the construction of the integral, but also in the resulting set of integrable functions. Indeed, Theorem 4.1 is the generalization of Theorem 2.7 in their paper to random integrands in the case where $S = \mathbb{R} \times E$.

Also observe that time or a filtration plays no role in the above definition of a Lévy basis. But we can obtain a random measure in the sense of Definition 2.1 in the following way: Let $S := \mathbb{R} \times E$ and choose a sequence $(E_k)_{k \in \mathbb{N}}$ in \mathcal{E} increasing to E . Then set

$\mathcal{E}_\Lambda := \bigcup_{k=1}^\infty \mathcal{E}|_{E_k}$ and define \mathcal{S} as the ring generated by the sets of the form $I \times U$, where $I = (s, t]$ with $-\infty < s < t < \infty$ and $U \in \mathcal{E}_\Lambda$. Now if Λ is a Lévy basis on (S, \mathcal{S}) , it induces a natural filtration $(\mathcal{F}_t)_{t \in \mathbb{R}}$ as follows:

\mathcal{F}_t is the completion of $\sigma(\Lambda(A) : A \in \mathcal{S}, A \subseteq (-\infty, t] \times E)$, $t \in \mathbb{R}$.

Then we can associate a random measure $\tilde{\Lambda}$ in the sense of Definition 2.1 to the Lévy basis Λ in the following way:

$$\tilde{\Lambda}(F \times (s, t] \times U) := 1_F \Lambda((s, t] \times U), \quad -\infty < s < t < \infty, U \in \mathcal{E}_\Lambda, F \in \mathcal{F}_s. \quad (6.2)$$

Setting $\tilde{O}_k := \Omega \times (-k, k] \times E_k$ for $k \in \mathbb{N}$, (6.2) allows for a unique extension of $\tilde{\Lambda}$ to $\tilde{\mathcal{P}}_{\tilde{\Lambda}} := \bigcup_{k \in \mathbb{N}} \tilde{\mathcal{P}}|_{\tilde{O}_k}$. This extension satisfies $\tilde{\Lambda}(\Omega \times A) = \Lambda(A)$ for any set $A \in \mathcal{S}$.

Comparing (6.1) with Theorem 3.3, it is natural to ask how the canonical decomposition of the random measure $\tilde{\Lambda}$ is related to the characteristics of the Lévy basis Λ . Let us assume from now on that Λ has no fixed times of discontinuity. Then recalling the construction in the proof of Theorem 3.3 and using [Sat04], Theorem 3.2, and [JS03], Chapter II, Theorem 4.15, we conclude that we may identify the characteristic triplets of $\tilde{\Lambda}$ with those of Λ . In particular, the canonical decomposition of $\tilde{\Lambda}$ determines the Lévy-Itô decomposition of Λ (see also [Ped03]).

From the considerations above it makes sense to identify the Lévy basis Λ with the random measure $\tilde{\Lambda}$ associated to it by (6.2), which we will do for the subsequent examples.

6.2 Ambit processes

As already discussed in the Section 1, **ambit processes** are one of the key examples where the stochastic integration theory presented in the sections above is needed. In this Section we cover the definition, existence conditions and stationarity of the general ambit process, in the following two sections we illustrate the results with concrete examples. So let $E = \mathbb{R}^d$ and Λ be a Lévy basis on $\mathbb{R} \times \mathbb{R}^d$ with no fixed time of discontinuity. Then consider the stochastic space-time model given by

$$Y(t, x) := \int_{A(t, x)} h(t, x, s, \xi) \sigma(s, \xi) \Lambda(ds, d\xi) + \int_{D(t, x)} q(t, x, s, \xi) a(s, \xi) d\xi ds, \quad t \in \mathbb{R}, x \in \mathbb{R}^d \quad (6.3)$$

where

- $h: \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ and $q: \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ are deterministic measurable functions,
- $\sigma: \tilde{\Omega} \rightarrow \mathbb{R}$ is predictable and $a: \tilde{\Omega} \rightarrow \mathbb{R}$ is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable,
- $A(t, x)$ and $D(t, x)$ are elements of $\mathcal{B}(\mathbb{R}^{d+1})$ for each $(t, x) \in \mathbb{R}^{d+1}$,

such that the integrals in (6.3) exist in the sense of (2.3). As an immediate consequence of (4.1), we have

Corollary 6.1 With the notation of (6.1), the ambit process Y is well-defined if and only if for every $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, the following conditions hold a.s.:

- (1) $\int_{A(t,x)} \left| h(t, x, s, \xi) \sigma(s, \xi) b(s, \xi) + \int_{\mathbb{R}} [\tau(h(t, x, s, \xi) \sigma(s, \xi) y) - h(t, x, s, \xi) \sigma(s, \xi) \tau(y)] \kappa(s, \xi, dy) \right| \lambda(ds, d\xi) < \infty,$
- (2) $\int_{A(t,x)} h^2(t, x, s, \xi)^2 \sigma^2(s, \xi) c(s, \xi) \lambda(ds, d\xi) < \infty,$
- (3) $\int_{A(t,x)} (1 \wedge (h(t, x, s, \xi) \sigma(s, \xi) y)^2) \kappa(s, \xi, dy) \lambda(ds, d\xi) < \infty,$
- (4) $\int_{D(t,x)} q(t, x, s, \xi) a(s, \xi) d\xi ds < \infty.$

Of particular interest are the cases where the ambit process is stationary in time, in space or in space and time. In the following, we only consider the first integral in (6.3), and by stationary we always means strictly stationary. We have the following theorem which we state without proof since it only requires standard methods. Recall the definition given in (2.6) and note that for some set $A \in \mathcal{P}$, $u \in \mathbb{R}$ and $z \in \mathbb{R}^d$ we write $A + (u, z) := \{(\omega, t + u, x + z) : (\omega, t, x) \in A\}.$

Theorem 6.2 For $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ let $Y(t, x)$ be given by

$$Y(t, x) := \int_{(S+t) \times \Xi(x)} h(t - s, x, \xi) \sigma(s, x) \Lambda(ds, d\xi) \quad (6.4)$$

$$\left(\text{resp. } Y(t, x) := \int_{S(t) \times (\Xi+x)} h(t, s, x - \xi) \sigma(s, \xi) \Lambda(ds, d\xi), \quad (6.5) \right.$$

$$\left. Y(t, x) := \int_{(t,x)+A} h(t - s, x - \xi) \sigma(s, \xi) \Lambda(ds, d\xi) \right), \quad (6.6)$$

where $S, S(t) \in \mathcal{B}(\mathbb{R})$, $\Xi, \Xi(x) \in \mathcal{B}(\mathbb{R}^d)$ and $A \in \mathcal{B}(\mathbb{R}^{d+1})$. Further assume that $\sigma \Lambda$ has stationary increments in time (resp. space, space and time), i.e.:

$$\begin{aligned} & ((\sigma \Lambda)(A_1), \dots, (\sigma \Lambda)(A_n)) \stackrel{d}{=} ((\sigma \Lambda)(A_1 + (u, 0)), \dots, (\sigma \Lambda)(A_n + (u, 0))) \\ & (\text{resp. } ((\sigma \Lambda)(A_1), \dots, (\sigma \Lambda)(A_n)) \stackrel{d}{=} ((\sigma \Lambda)(A_1 + (0, z)), \dots, (\sigma \Lambda)(A_n + (0, z))), \\ & ((\sigma \Lambda)(A_1), \dots, (\sigma \Lambda)(A_n)) \stackrel{d}{=} ((\sigma \Lambda)(A_1 + (u, z)), \dots, (\sigma \Lambda)(A_n + (u, z))) \quad) \end{aligned}$$

for all $A_1, \dots, A_n \in \tilde{\mathcal{P}}_{\sigma \Lambda}$, $u > 0$ and $z \in \mathbb{R}^d$. Then Y is stationary in time (resp. space, space and time).

In particular, the statement holds if σ is stationary in time (resp. space, space and time), Λ is independent of σ and the triplet (b, c, κ) does not depend on t (resp. x , (t, x)) with $\lambda(dt, dx) = dt \pi(dx)$ (resp. $\alpha(dt) d\xi, dt d\xi$) and some σ -finite measures α on \mathbb{R} and π on \mathbb{R}^d .

6.3 The supCOGARCH

Let us briefly recall the definition of the COGARCH process (see [KLM04]). For simplicity we directly start with the two-sided stationary version: Let $(S_t)_{t \in \mathbb{R}}$ be a two-sided subordinator with no drift and Lévy measure ν_S . For $\varphi, \eta > 0$ define another two-sided Lévy process X^φ by

$$X_t^\varphi := \eta t - \sum_{0 < s \leq t} \log(1 + \varphi \Delta S_s), \quad t \geq 0, \quad X_t^\varphi := \eta t + \sum_{t < s \leq 0} \log(1 + \varphi \Delta S_s), \quad t < 0.$$

Then the two-sided stationary COGARCH V^φ is defined as

$$V_t^\varphi := \beta \int_{-\infty}^t e^{-(X_t^\varphi - X_s^\varphi)} ds, \quad t \in \mathbb{R}, \quad (6.7)$$

where $\beta > 0$ is a constant. By [KLM04], Theorem 3.1, the integral in (6.7) converges a.s. for every $t \in \mathbb{R}$ if and only if φ satisfies

$$\int_{\mathbb{R}_+} \log(1 + \varphi y) \nu_S(dy) < \eta. \quad (6.8)$$

Hence let us denote the collection of all $\varphi > 0$ satisfying (6.8) by Φ . From (6.8) we see that Φ must be of the form $(0, \varphi_{\max})$ with some $0 < \varphi_{\max} < \infty$.

From [KLM04], Proposition 2.1, we also know that V^φ satisfies the following stochastic differential equation:

$$dV_t^\varphi = (\beta - \eta V_t^\varphi) dt + \varphi V_{t-}^\varphi dS_t, \quad t \in \mathbb{R}. \quad (6.9)$$

In [BCK13], several approaches to construct superpositions of COGARCH processes with different values of φ are suggested in order to obtain different jump size scales in the resulting processes. With β and η remaining constant, one possibility is to take a Lévy basis Λ with characteristics $(0, 0, dt \nu_S(dy) \pi(d\varphi))$ w.r.t. the truncation function $\tau = 0$, where π is a probability measure on Φ and ν_S a Lévy measure with $\nu_S((-\infty, 0]) = 0$. Setting

$$S_t := \Lambda((0, t], \Phi), \quad t \geq 0, \quad S_t := -\Lambda((t, 0], \Phi), \quad t < 0,$$

we can define the COGARCH process V^φ for each $\varphi \in \Phi$ by (6.7). Motivated by (6.9), the **supCOGARCH** \bar{V} is now defined by the integral equation

$$\bar{V}_t = \bar{V}_0 + \int_0^t (\beta - \eta \bar{V}_s) ds + \int_0^t \int_\Phi \varphi V_{s-}^\varphi \Lambda(ds, d\varphi), \quad t \geq 0, \quad (6.10)$$

with some random variable \bar{V}_0 which is independent of Λ .

Using [Pro05], Chapter V, Theorem 7, and Itô's formula for semimartingales (cf. [JS03], Chapter I, Theorem 4.57), one can easily check that this integral equation has the unique solution

$$\bar{V}_t = e^{-\eta t} \left(\bar{V}_0 + \int_0^t e^{\eta s} dA_s + \beta \int_0^t e^{\eta s} ds \right), \quad t \geq 0, \quad (6.11)$$

where A is given by

$$A_t := \int_0^t \int_{\Phi} \varphi V_{s-}^{\varphi} \Lambda(ds, d\varphi), \quad t \geq 0, \quad A_t := - \int_t^0 \int_{\Phi} \varphi V_{s-}^{\varphi} \Lambda(ds, d\varphi), \quad t < 0.$$

In [BCK13] it is shown that the supCOGARCH \bar{V} as in (6.11) becomes strictly stationary whenever \bar{V}_0 satisfies

$$\bar{V}_0 \stackrel{d}{=} \frac{\beta}{\eta} + \int_0^{\infty} e^{-\eta s} dA_s = \frac{\beta}{\eta} + \int_0^{\infty} \int_{\Phi} e^{-\eta s} \varphi V_{s-}^{\varphi} \Lambda(ds, d\varphi). \quad (6.12)$$

A necessary and sufficient condition for the existence of the right-hand side can be derived from Theorem 4.1 and Remark 4.3:

Corollary 6.3 The integral on the right-hand side of (6.12) exists if and only if

$$\int_{\mathbb{R}_+} \int_{\Phi} \int_{\mathbb{R}_+} 1 \wedge (y \varphi e^{-\eta s} V_s^{\varphi}) \nu_S(dy) \pi(d\varphi) ds < \infty \quad \text{a.s.} \quad (6.13)$$

In [BCK13] one can find some sufficient conditions for (6.13) to hold, which are easier to verify. Now if we assume (6.13) and make the specific choice

$$\bar{V}_0 := \frac{\beta}{\eta} + \int_{-\infty}^0 e^{\eta s} dA_s,$$

we can rewrite (6.11) and obtain

$$\bar{V}_t = \int_{-\infty}^t e^{-\eta(t-s)} dA_s + \beta \int_{-\infty}^t e^{-\eta(t-s)} ds = \frac{\beta}{\eta} + \int_{-\infty}^t \int_{\Phi} e^{-\eta(t-s)} \varphi V_{s-}^{\varphi} \Lambda(ds, d\varphi), \quad t \in \mathbb{R},$$

such that \bar{V} is an ambit process as in (6.4) up to a constant. In particular, the supCOGARCH \bar{V} provides an example where the stochastic volatility process $\sigma(s, \varphi) := \varphi V_{s-}^{\varphi}$ is *not* independent of the underlying Lévy basis Λ .

In [BCK13], properties of \bar{V} such as stationarity and jump behaviour are analysed. As already mentioned, one can also find alternative ways there to define a superposition of COGARCH processes which differ from the one discussed above in construction and behaviour.

6.4 Volatility modulated supOU and supCARMA processes

After supOU processes have been suggested in [BN01] in order to introduce a long memory effect in Ornstein-Uhlenbeck processes, a multivariate version has been proposed in [BNS11]. Let us recall their definition: let $E := M_d^-$ denote the set of all matrices in $\mathbb{R}^{d \times d}$ whose eigenvalues all have a strictly negative real part and let Λ be a d -dimensional Lévy basis on $\mathbb{R} \times M_d^-$ with characteristics given by $(b dt \pi(dA), c dt \pi(dA), \nu(dy) dt \pi(dA))$, where π is a probability measure on M_d^- . The d -dimensional **supOU process** is then defined as

$$X_t := \int_{-\infty}^t \int_{M_d^-} e^{A(t-s)} \Lambda(ds, dA), \quad t \in \mathbb{R}. \quad (6.14)$$

X is stationary, has an infinitely divisible distribution and exists under the conditions given in [BNS11], Theorem 3.1 (or see Corollary 6.4 below). Further results such as existence of moments, second order structure or long-range dependence can also be found in [BNS11].

As suggested in [BNV12] (see Section 5, in particular), it is reasonable to implement a volatility modulation in (6.14), which leads to

$$Y_t := \int_{-\infty}^t \int_{M_d^-} e^{A(t-s)} \sigma(s, A) \Lambda(ds, dA), \quad t \in \mathbb{R}, \quad (6.15)$$

where $\sigma: \Omega \times \mathbb{R} \times M_d^- \rightarrow \mathbb{R}$ is a $\mathcal{P} \otimes \mathcal{B}(M_d^-)$ -measurable function. Note that in contrast to [BNV12], we do not require σ and Λ to be independent such that we may, for instance, model σ itself (or a function of σ) as an ambit process:

$$\sigma(t, A) := \int_{(S+t-)\times\Xi(A)} g(t-s-, A, \bar{A}) \bar{\Lambda}(ds, d\bar{A}), \quad t \in \mathbb{R}, A \in M_d^-,$$

where g is a measurable function, $\Xi(A) \subset M_d^-$ for all $A, S \in \mathcal{B}(\mathbb{R})$ and $\bar{\Lambda}$ is another Lévy basis, which in general is *not* independent of Λ .

From Theorem 4.9 we can obtain necessary and sufficient conditions for (6.15) to exist. If σ is stationary in time, they are as follows:

Corollary 6.4 In the stationary case, the volatility modulated supOU process as in (6.15) exists if and only if a.s.,

- (1) $\int_0^\infty \int_{M_d^-} \left\| e^{As} b \sigma(s, A) + \int_{\mathbb{R}^d} [\tau(e^{As} y \sigma(s, A)) - e^{As} \tau(y) \sigma(s, A)] \nu(dy) \right\| dt \pi(dA) < \infty,$
- (2) $\int_0^\infty \int_{M_d^-} \sigma^2(s, A) e^{As} c(e^{As})^T dt \pi(dA) < \infty,$
- (3) $\int_0^\infty \int_{M_d^-} \int_{\mathbb{R}^d} (1 \wedge (\sigma(s, A) \|e^{As} y\|)^2) \nu(dy) dt \pi(dA) < \infty,$

where $\|\cdot\|$ is an arbitrary norm on \mathbb{R}^d .

Closely related to multivariate OU and supOU processes are the multivariate causal CARMA and supCARMA processes, which play an important role in time series modelling (see [Bro01, MS07]). As in [MS07], Definition 3.20, let us fix a d -dimensional Lévy process $(L_t)_{t \in \mathbb{R}}$, numbers $p, q \in \mathbb{N}_0$ with $q < p$, matrices $A_1, \dots, A_p, B_0, \dots, B_q \in \mathbb{R}^{d \times d}$ with $B_0 \neq 0$. Further assume that

$$A := \begin{pmatrix} 0 & I_d & 0 & \cdots & 0 \\ 0 & 0 & I_d & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & I_d \\ -A_p & -A_{p-1} & \cdots & \cdots & -A_1 \end{pmatrix} \in \mathbb{R}^{dp \times dp} \quad (6.16)$$

only has eigenvalues with strictly negative real part, and define the $(dp \times d)$ -matrix $\beta = (\beta_1, \dots, \beta_p)^T$ by

$$\beta_{p-j} := - \sum_{i=1}^{p-j-1} A_i \beta_{p-j-i} + B_{q-j}, \quad j = 0, \dots, q, \quad \beta_1 := \dots := \beta_{p-q-1} := 0, \quad \text{if } p > q + 1. \quad (6.17)$$

Then the d -dimensional causal CARMA process defined by the following integral:

$$Y_t := (I_d, 0, \dots, 0) \int_{-\infty}^t e^{A(t-s)} \beta \, dL_s, \quad t \in \mathbb{R}. \quad (6.18)$$

Conditions for this integral to exist can be found in [MS07] or in Corollary 6.5 below.

From here it is clear how a superposition of CARMA processes (with or without volatility modulation) can be constructed (see [BNS11], Section 5.3). The numbers p, q with $p > q$ remain fixed throughout. Let \mathcal{A} be the collection of all matrices in M_{dp}^- of the form (6.16) and \mathcal{B} be the collection of all $(dp \times d)$ -matrices of the form (6.17). With $E := \mathcal{A} \times \mathcal{B}$ let Λ be a d -dimensional Lévy basis on $\mathbb{R} \times E$ with characteristics

$$(b \, dt \, \pi(dA) \rho(d\beta), c \, dt \, \pi(dA) \rho(d\beta), \nu(dy) \, dt \, \pi(dA) \rho(d\beta)),$$

where π and ρ are probability measures on \mathcal{A} and \mathcal{B} , respectively. Furthermore, fix a $\tilde{\mathcal{P}}$ -measurable function $\sigma: \tilde{\Omega} \rightarrow \mathbb{R}$. Then the d -dimensional **volatility modulated sup-CARMA process** is defined by

$$\bar{Y}_t = (I_d, 0, \dots, 0) \int_{-\infty}^t \int_{\mathcal{A}} \int_{\mathcal{B}} e^{A(t-s)} \beta \sigma(s, A, \beta) \Lambda(ds, dA, d\beta), \quad t \in \mathbb{R}. \quad (6.19)$$

Necessary and sufficient conditions for \bar{Y} to exist can again be derived from Theorem 4.9. In the case where σ is stationary in time, these are given by

Corollary 6.5 In the stationary case, \bar{Y} is well-defined if and only if a.s.:

- (1) $\int_0^\infty \int_{\mathcal{A}} \int_{\mathcal{B}} \left\| Ie^{As} \beta b \sigma(s, A, \beta) + \int_{\mathbb{R}^d} [\tau(Ie^{As} \beta y \sigma(s, A, \beta)) - Ie^{As} \beta \tau(y) \sigma(s, A, \beta)] \nu(dy) \right\| dt \pi(dA) \rho(d\beta) < \infty,$
- (2) $\int_0^\infty \int_{\mathcal{A}} \int_{\mathcal{B}} \sigma^2(s, A, b) Ie^{As} \beta c (Ie^{As} \beta)^T dt \pi(dA) \rho(d\beta) < \infty,$
- (3) $\int_0^\infty \int_{\mathcal{A}} \int_{\mathcal{B}} \int_{\mathbb{R}^d} (1 \wedge (\sigma(s, A, b) \|Ie^{As} \beta y\|)^2) \nu(dy) dt \pi(dA) \rho(d\beta) < \infty,$

where $I = (I_d, 0, \dots, 0)$ and $\|\cdot\|$ is an arbitrary norm on \mathbb{R}^d .

Acknowledgement

We take pleasure in thanking Jean Jacod for extremely useful discussions and his advice on this topic. The first author acknowledges support from the graduate program TopMath of the Elite Network of Bavaria and the TopMath Graduate Center of TUM Graduate School at Technische Universität München.

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